Constructing the complex numbers

Let $\mathbb{R}$ denote the real numbers, and $\mathbb{R}[X]$ the ring of polynomials with real coefficients. Then we have the subring $\mathbb{C} := \mathbb{R}[X]/(X^2+1)\mathbb{R}[X]$. In $\mathbb{C}$, we have:

$$X^2 + 1 = 0 \;\Leftrightarrow\; X^2 = -1$$

$\mathbb{R}[X]$ is Euclidean, hence $\mathbb{C} = \{a + bX + (X^2+1)\mathbb{R}[X] \mid a, b \in \mathbb{R}\}$.

Let $0 \neq a + bX \in \mathbb{C}$. Then:

$$(a+bX) \cdot \left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} X \right) = 1$$

Thus $\mathbb{C}$ is a field where -1 is a square.