HOMOTOPY GROUPS OF OPERATOR GROUPS

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Abstract. In this text we summarize some of the results of [Nee02, Sec. II]. More precisely, we will first show in Section 2 that $\mathrm{GL}(H)$ is contractible for infinite-dimensional H; this is Kuiper's theorem. We then use this result in Section 3 in order to prove that several other classical operator groups are contractible. In Section 4 we recall some results of Palais regarding the topology of infinite-dimensional vector spaces; these are then used to compute the homotopy groups of infinite matrix groups (in Sec. 5) and of congruence subgroups for the Schatten ideals $B_p(H)$ (in Sec. 6).

Notation. We consider real and complex Hilbert spaces ([Nee02] also handles the quaternionic case). The Banach spaces of bounded and compact operators on H are denoted by $\mathrm{B}(H)$ and $\mathrm{K}(H)$, respectively, and the space of Hermitian (i.e. self-adjoint) operators is denoted by $\mathrm{Herm}(H)$. We write $\mathrm{GL}(H)$ and $\mathrm{U}(H)$ for the invertible and unitary operators, respectively.

Linear Banach-Lie groups. Recall that both $\mathrm{GL}(H)$ and $\mathrm{U}(H)$ are *Banach-Lie groups*. Their respective Lie algebras are given by

$$\mathfrak{gl}(H) := \mathrm{B}(H)$$
$$\mathfrak{u}(H) := \{X \in \mathfrak{gl}(H) : X^* = -X\}$$

(cf. [Nee06]). Furthermore we have a *polar decomposition* implemented by the diffeomorphism

$$\mathrm{U}(H) \times \mathrm{Herm}(H) \to \mathrm{GL}(H), (u, X) \mapsto ue^X$$

In particular, $\mathrm{U}(H)$ is a deformation retract of $\mathrm{GL}(H)$.

1. Kuiper's theorem

In this section we want to prove the following theorem.

Theorem 1 (Kuiper's theorem)**.** $\mathrm{GL}(H)$ *is contractible for every infinite-dimensional Hilbert space* H*.*

The proof for separable H can be found in [Kui65]; thus we will only consider the inseparable case. The following theorem due to Palais shows that in fact it will be sufficient to show that all maps $\mathbb{S}^k \to \mathrm{GL}(H)$ are homotopic to a constant map.

Theorem 2. *A metrizable topological manifold modeled over a sequentially complete locally convex space is contractible if and only if all homotopy groups vanish.*

Proof. [Pal66, Cor. to Thm. 15]. □

The following lemma allows us to decompose any Hilbert space into the direct sum of copies of l^2; this will turn out to be rather convenient in what follows.

Lemma 3. *Let H be a Hilbert space, $M \subseteq \mathrm{B}(H)$ a separable set of operators. Then there exists an orthogonal decomposition*

$$H \cong \bigoplus^{\perp} H_j$$

into closed, separable, M-invariant subspaces $(H_j)_{j\in J}$.

If H is infinite-dimensional, the H_i can be chosen to be infinite-dimensional as well, so that

$$H \cong l^2(J, l^2(\mathbb{N}, \mathbb{K}))$$

Proof. (1) We may assume w.l.o.g. that $M = M^* \ni 1$. Zorn's lemma yields a maximal set $(H_j)_{j\in J}$ of non-zero, pairwise orthogonal, closed, separable, M-invariant subspaces of H. Let $H_0 := \overline{\sum H_j}$.

Assume $H_0 \neq H$. Since H_0 is $M^{(*)}$-invariant, $H_0^\perp$ is $M^{(*)}$-invariant. Thus for any $0 \neq v \in H_0^\perp$ the closed, separable, M-invariant subspace $H_\infty := \overline{\mathrm{span}(Mv)}$ is orthogonal to the H_j, contradicting maximality.

(2) Now assume that H is infinite-dimensional. Consider

$$J_0 := \{j \in J : \dim H_j < \infty\}$$

If J_0 is finite, there is some $j \in J \backslash J_0$ and we can simply append the finitely-many finite-dimensional subspaces to H_j.

If J_0 is infinite, then $\#J_0 = \#(J_0 \times \mathbb{N})$ and J_0 can be decomposed into (infinitely many) countably infinite sets.. Thus we can replace the finite summands by infinite-dimensional separable ones. □

The following proposition concludes the proof of Kuiper's theorem.

Proposition 4. *If X is a separable topological space and H is an inseparable Hilbert space, then every continuous map $f : X \to \mathrm{GL}(H)$ is homotopic to a constant map.*

Proof. (1) The main ingredient of the proof is the following "trick": For every $x \in \mathrm{GL}(H)$, we have a path

$$[0,1] \to \mathrm{GL}(H^2), t \mapsto \begin{pmatrix} 1 & 0 \\ t(x^{-1}-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t(x-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & -tx^{-1} \\ 0 & 1 \end{pmatrix}$$

connecting 1 and $\mathrm{diag}(x, x^{-1})$.

(2) Since $f(X)$ is a separable set of operators, the lemma yields

$$H \cong l^2(J, l^2(\mathbb{N}, \mathbb{K}))$$

such that the operators act diagonally (on the "outer" l^2).

Since H is inseparable, the index set J must be (uncountably) infinite. Thus we can decompose $J = J_1 \dot\cup J_2$ into disjoint sets of equal cardinality $\#J = \#J_1 = \#J_2$ which in turn leads to an orthogonal decomposition $H \cong H \oplus H$ which $f(X)$ acts diagonally on. Thus we can regard f as a map

$$f = \mathrm{diag}(g_1, g_2) \overset{(1)}{\simeq} \mathrm{diag}(g_1 g_2, 1) =: \mathrm{diag}(g, 1) =: \tilde{f}$$

(3) We need to create some more space before we can finish the proof. Since $\#J = \#(J \times \mathbb{N})$, we can decompose the (second summand) H further as follows:

$$H \cong H \oplus H \cong H \oplus l^2(\mathbb{N}, H)$$

In this picture, $\tilde{f}$ corresponds to the map

$$\tilde{f} = \mathrm{diag}(g,1,1,\ldots) = \mathrm{diag}(g,1,1,\ldots)\ \underbrace{\mathrm{diag}(1,1,1,\ldots)}_{\overset{(1)}{\simeq}\mathrm{diag}(g^{-1},g,g^{-1},\ldots)}$$

$$\simeq \mathrm{diag}(1,g,g^{-1},\ldots) = \mathrm{diag}(1,g,g^{-1},\ldots)\ \underbrace{\mathrm{diag}(1,1,1,\ldots)}_{\overset{(1)}{\simeq}\mathrm{diag}(1,g^{-1},g,g^{-1},\ldots)} \simeq 1$$

□

Corollary 5. $\mathrm{U}(H)$ *is contractible for every infinite-dimensional Hilbert space* H.

2. Contractibility of other classical linear Lie groups

In this section H will denote a *complex* Hilbert space with a *conjugation* I, i.e. an antilinear isometry with $I^2 = 1$.

Then we can consider the groups

$$\mathrm{GL}(H,I) := \{g \in \mathrm{GL}(H) : g^{-1} = Ig^*I^{-1}\}$$
$$\mathrm{U}(H,I) := \mathrm{GL}(H,I) \cap \mathrm{U}(H)$$

Example 6. Complex conjugation $\bar{\cdot}$ is a conjugation in $L^2 := L^2(\Omega,\mathbb{C})$. In that case,

$$\mathrm{GL}(L^2,\bar{\cdot}) = \{g \in \mathrm{GL}(L^2) : \overline{g^{-1}f} = g^*\overline{f} \quad (\forall f \in L^2)\}$$
$$\mathrm{U}(L^2,\bar{\cdot}) = \{g \in \mathrm{U}(L^2) : \overline{g^*f} = g^*\overline{f} \quad (\forall f \in L^2)\}$$

For instance, $f \mapsto -f \in \mathrm{U}(L^2,\bar{\cdot})$.

It follows from Kuiper's theorem that these groups are contractible as well. More precisely, we have the following results.

Proposition 7. *We have*

$$\mathrm{U}(H,I) \cong \mathrm{U}(H^I_{\mathbb{R}})$$

where $H^I := \{x \in H : Ix = x\}$.

In particular, $\mathrm{U}(H,I)$ *is contractible for infinite-dimensional* H.

Proof. Consider the continuous group homomorphism

$$\mathrm{U}(H,I) \to \mathrm{U}(H^I_{\mathbb{R}}), u \mapsto u|_{H^I}$$

which is well-defined since every element in $\mathrm{U}(H,I)$ commutes with I. The relation $H = H^I \oplus iH^I$ now shows how to construct a continuous inverse. □

Proposition 8. *We have a polar decomposition*

$$\mathrm{GL}(H,I) \cong \mathrm{U}(H,I) \times \mathrm{Herm}(H,I)$$

with $\mathrm{Herm}(H,I) := \{X \in \mathrm{Herm}(H) : X = -IX^*I^{-1}\}$.

Thus $\mathrm{GL}(H,I)$ *is contractible for infinite-dimensional* H.

Proof. Let

$$\tau \in \mathrm{Aut}(\mathrm{GL}(H)), g \mapsto I(g^*)^{-1}I^{-1}$$
$$\tau_{\mathfrak{g}} \in \mathrm{Aut}(\mathfrak{gl}(H)), X \mapsto -IX^*I^{-1}$$

Then $\mathrm{GL}(H,I) = \mathrm{GL}(H)^\tau$ and $\tau(g) = \tau(u)e^{\tau_{\mathfrak{g}}(X)}$ is the *unique* polar decomposition of $g = ue^X \in \mathrm{GL}(H)$. Thus $\tau(g) = g$ if and only if $u \in \mathrm{U}(H,I)$ and $x \in \mathrm{Herm}(H,I)$.

Consequently, the polar decomposition in $\mathrm{GL}(H)$ restricts to the desired polar decomposition for $\mathrm{GL}(H, I)$, and contractibility follows from the preceding lemma. □

See [Nee02, Sec. II.2] for a treatment of other classical linear Lie groups such as those arising from *anti*conjugations ($I^2 = -1$).

3. Topology of infinite-dimensional vector spaces

The following results are also due to Palais.

Theorem 9. *(i) Let X be a locally convex topological vector space and $E \subseteq X$ a dense subspace endowed with the direct limit topology with respect to the finite-dimensional subspaces. If $U \subseteq X$ is an open subset and $U \cap E$ is considered with the subspace topology in E, then the continuous inclusion $U \cap E \hookrightarrow U$ is a weak homotopy equivalence.*

(ii) Let $f : X \to Y$ be a morphism between metrizable locally convex topological vector spaces and $U \subseteq Y$ open. Then $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a homotopy equivalence.

Proof. [Pal66, Thm. 12 and 16]. □

Lemma 10. *Let E be a real vector space endowed with the direct limit topology with respect to its finite-dimensional subspaces. Then the following assertions hold:*

(i) Each linearly independent subset is closed and discrete.
(ii) Each compact subset is contained in a finite-dimensional subspace.
(iii) For each subset $U \subseteq E$ and $u_0 \in U$ we have

$$\pi_k(U, u_0) \cong \lim_{F \in \mathcal{F}} \pi_k(U \cap F, u_0)$$

where $\mathcal{F}$ denotes the directed set of all finite-dimensional spaces $F \subseteq E$ containing u.

Proof. (i) Every linearly-independent subset $S \subseteq E$ is closed since its intersection with every finite-dimensional subspace is closed (even finite). By the same argument, every subset of S is closed; hence S is discrete.

(ii) Suppose $C \subseteq E$ is compact. Take a maximal linearly independent subset $S \subseteq C$. By (i), S is compact and discrete, hence finite. Thus C is contained in the finite-dimensional subspace $\operatorname{span} S$.

(iii) By (ii), the image of any continuous map $(\mathbb{S}^k, 1) \to (U, u_0)$ is contained in a finite-dimensional subspace $F \subseteq E$. It follows that the natural homomorphism

$$\lim_{F \in \mathcal{F}} \pi_k(U \cap F, u_0) \to \pi_k(U, u_0)$$

is surjective. The same argument also shows injectivity since every homotopy has compact domain. □

4. Homotopy groups of the stable matrix groups

The *matrix algebra* with index set J is defined as

$$\mathrm{M}(J, \mathbb{K}) := \{(m_{i,j}) \in \mathbb{K}^{J \times J} : \text{ only finitely many } m_{i,j} \neq 0\}$$

It is unital if and only if J is finite. The *group of invertible matrices* is then given by

$$\mathrm{GL}(J, \mathbb{K}) := (1 + \mathrm{M}(J, \mathbb{K}))^\times$$

For $F \subseteq J$ we have natural identifications $\mathrm{M}(F,\mathbb{K}) \subseteq \mathrm{M}(J,\mathbb{K})$ and $\mathrm{GL}(F,\mathbb{K}) \subseteq \mathrm{GL}(J,\mathbb{K})$. It follows that

$$\mathrm{M}(J,\mathbb{K}) = \varinjlim \mathrm{M}(F,\mathbb{K})$$
$$\mathrm{GL}(J,\mathbb{K}) = \varinjlim \mathrm{GL}(F,\mathbb{K})$$

This holds even if we only consider *finite* subsets $F \subseteq J$, which is what we will do now. Then there are natural topologies on the $\mathrm{M}(F,\mathbb{K})$ and $\mathrm{GL}(F,\mathbb{K})$. Thus we endow $\mathrm{M}(J,\mathbb{K})$ and $\mathrm{GL}(J,\mathbb{K})$ with the respective final topologies so that the above direct limits can also be understood in the topological sense.

Note that in general multiplication will *not* be (jointly) continuous (but left- and right- multiplication will always be).

Proposition 11. *For every $k \in \mathbb{N}_0$ we have*

$$\pi_k(\mathrm{M}(J,\mathbb{K})) = \varinjlim \pi_k(\mathrm{M}(F,\mathbb{K}))$$
$$\pi_k(\mathrm{GL}(J,\mathbb{K})) = \varinjlim \pi_k(\mathrm{GL}(F,\mathbb{K}))$$

Proof. This follows from Lemma 10 (iii). ☐

Note that we recover the familiar matrix algebras and groups for $J = \{1,\dots,n\}$ (together with their natural topology).

Proposition 12. *Every injection $\mathbb{N} \hookrightarrow J$ induces a weak homotopy equivalence $\mathrm{GL}(\mathbb{N},\mathbb{K}) \hookrightarrow \mathrm{GL}(J,\mathbb{K})$.*

Proof. We can assume w.l.o.g. that $\mathbb{N} \subseteq J$.

(1) Suppose $F, \tilde{F} \subseteq J$ are finite disjoint subsets with equal cardinality. Then using the same "trick" as in the proof of Proposition 4 we see that every continuous map $X \to \mathrm{GL}(F,\mathbb{K})$ is homotopic in $\mathrm{GL}(J,\mathbb{K})$ to a continuous map $X \to \mathrm{GL}(\tilde{F},\mathbb{K})$.

(2) Surjectivity: Let $[f] \in \pi_k(\mathrm{GL}(J,\mathbb{K}))$. In view of Lemma 10 (ii) the image of f is contained in some $\mathrm{GL}(F,\mathbb{K})$ for finite $F \subseteq J$. But by part (1) we can homotope f to a map with image in $\mathrm{GL}(\mathbb{N},\mathbb{K})$; this is a preimage.

(3) Injectivity: Let $[f] \in \ker(\pi_k(\mathrm{incl}))$, i.e. there is a homotopy H between f and the constant map 1 in $\mathrm{GL}(J,\mathbb{K})$. Again by compactness, the image of H is contained in some $\mathrm{GL}(F,\mathbb{K})$ for finite $F \subseteq J$. Thus it follows from $\mathrm{GL}(F,\mathbb{K}) \cong \mathrm{GL}(\#F,\mathbb{K}) \subseteq \mathrm{GL}(\mathbb{N},\mathbb{K})$ that f is nullhomotopic already in $\mathrm{GL}(\mathbb{N},\mathbb{K})$. ☐

Corollary 13. *For every infinite J and $k \in \mathbb{N}_0$ we have*

$$\pi_k(\mathrm{GL}(\mathbb{N},\mathbb{K})) \cong \pi_k(\mathrm{GL}(J,\mathbb{K}))$$

The following classical results by Bott [Bot59] describe the homotopy groups of $\mathrm{GL}(\mathbb{N},\mathbb{K})$. In view of the preceding corollary they hold for arbitrary stable matrix groups $\mathrm{GL}(J,\mathbb{K})$, J infinite.

Theorem 14 (Stability)**.** *Let $k \in \mathbb{N}$. Then for $n \in \mathbb{N}$ large enough the maps $\mathrm{GL}(n,\mathbb{K}) \hookrightarrow \mathrm{GL}(n+1,\mathbb{K})$ induce isomorphisms*

$$\pi_k(\mathrm{GL}(n,\mathbb{K})) \xrightarrow{\cong} \pi_k(\mathrm{GL}(n+1,\mathbb{K}))$$

(the homotopy groups "stabilize") so that

$$\pi_k(\mathrm{GL}(\mathbb{N},\mathbb{K})) \cong \pi_k(\mathrm{GL}(n,\mathbb{K}))$$

Sketch of proof. Let $d := \dim \mathbb{K}$. The transitive action

$$\mathrm{U}(n+1,\mathbb{K}) \circlearrowleft \mathbb{S}^{d(n+1)-1}$$

leads to a locally trivial principal bundle

$$\mathrm{U}(n,\mathbb{K}) \hookrightarrow \mathrm{U}(n+1,\mathbb{K}) \to \mathbb{S}^{d(n+1)-1}$$

The long exact sequence for this fibration is given by

$$\ldots \to \pi_{k+1}(\mathbb{S}^{d(n+1)-1}) \to \pi_k(\mathrm{U}(n,\mathbb{K})) \longrightarrow \pi_k(\mathrm{U}(n+1,\mathbb{K})) \to \pi_k(\mathbb{S}^{d(n+1)-1}) \to \ldots$$

and the fact that the outer homotopy groups vanish for $k+1 < d(n+1)-1$ implies the first claim.

The second assertion now follows from 11. □

Theorem 15 (Bott Periodicity)**.** *We have the following periodicity relations*

$$\pi_k(\mathrm{GL}(\mathbb{N},\mathbb{C})) \cong \pi_{k+2}(\mathrm{GL}(\mathbb{N},\mathbb{C}))$$
$$\pi_k(\mathrm{GL}(\mathbb{N},\mathbb{R})) \cong \pi_{k+8}(\mathrm{GL}(\mathbb{N},\mathbb{R}))$$

so that we can determine the homotopy groups of $\mathrm{GL}(\mathbb{N},\mathbb{K})$ *from the following table:*

	$\mathrm{GL}(\mathbb{N},\mathbb{R})$	$\mathrm{GL}(\mathbb{N},\mathbb{C})$
π_0	$\mathbb{Z}/2\mathbb{Z}$	0
π_1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$
π_2	0	0
π_3	$\mathbb{Z}$	$\mathbb{Z}$
π_4	0	0
π_5	0	$\mathbb{Z}$
π_6	0	0
π_7	$\mathbb{Z}$	$\mathbb{Z}$

5. Homotopy groups of the congruence subgroups of the Schatten ideals

In this section, H is again a $\mathbb{K}$-Hilbert space. The *Schatten ideals* are the Banach spaces defined by

$$\mathrm{B}_p(H) := \{x \in \mathrm{B}(H) : ||x||_p < \infty\}$$
$$||x||_p := \Big(\mathrm{tr}((x^*x)^{p/2})\Big)^{1/p}$$

for $p \in [1,\infty)$. We also define

$$\mathrm{B}_\infty(H) := \mathrm{K}(H)$$
$$||\cdot||_\infty := ||\cdot||$$

They have the following properties.

Proposition 16 (cf. my Zwischentreffen talk)**.**

(i) *The* $\mathrm{B}_p(H)$ *are ideals in* $\mathrm{B}(H)$*.*
(ii) *We have*

$$\mathrm{B}_{fin}(H) \subseteq \mathrm{B}_1(H) \subseteq \mathrm{B}_p(H) \subseteq \mathrm{B}_q(H) \subseteq \mathrm{B}_\infty(H) \subseteq \mathrm{B}(H)$$

for $1 \leqslant p \leqslant q \leqslant \infty$*.*
(iii) *For any* $x = \sum a_j \langle\cdot, e_j\rangle f_j \in \mathrm{B}_\infty(H)$ *with ONB* (e_j)*,* (f_j) *we have*

$$||x||_p = ||(a_j)||_{l^p}$$

(iv) *If* (e_j) *is any ONB of* H*, the set of projections* $\{\langle\cdot, e_i\rangle e_j\}$ *is total in each of the spaces* $B_p(H)$*,* $p \in [1,\infty]$*.*

The *congruence subgroups* of the Schatten ideals and the corresponding unitaries are the Banach-Lie groups given by

$$\begin{aligned} \mathrm{GL}_p(H) &:= \mathrm{GL}(H) \cap (1 + \mathrm{B}_p(H)) \\ \mathrm{U}_p(H) &:= \mathrm{GL}_p(H) \cap \mathrm{U}(H) \end{aligned}$$

Their Lie algebras are given by

$$\begin{aligned} \mathfrak{gl}_p(H) &:= \mathrm{B}_p(H) \\ \mathfrak{u}_p(H) &:= \mathrm{B}_p(H) \cap \mathfrak{u}(H) \end{aligned}$$

respectively. Oonce again we have a polar decomposition

$$\mathrm{GL}_p(H) \cong \mathrm{U}_p(H) \times \mathrm{Herm}_p(H)$$

with $\mathrm{Herm}_p(H) := \mathrm{Herm}(H) \cap \mathrm{B}_p(H)$ (cf. [Nee00, Def. IV.20, Prop. A.4]).

Theorem 17. *Let H be an infinite-dimensional $\mathbb{K}$-Hilbert space and $p \in [1, \infty]$. Then the following assertions hold:*

(i) $\pi_k(\mathrm{GL}_p(H)) \cong \pi_k(\mathrm{GL}(\mathbb{N}, \mathbb{K}))$
(ii) The inclusion map $\mathrm{GL}_p(H_s) \hookrightarrow \mathrm{GL}_p(H)$ is a weak homotopy equivalence for every infinite-dimensional separable subspace $H_s \subseteq H$.
(iii) The inclusion map $\mathrm{GL}_p(H) \hookrightarrow \mathrm{GL}_q(H)$ is a homotopy equivalence for $p \leqslant q$.

Proof. (i) Fix an ONB (e_j). By the previous proposition, $\mathrm{B}_0(H) := \mathrm{span}\{\langle \cdot, e_i \rangle e_j\} \subseteq \mathrm{B}_p(H)$ is dense. We endow $\mathrm{B}_0(H)$ with the direct limit topology with respect to the directed set of its finite-dimensional subspaces.

Then there is a natural isomorphism $B_0(H) \cong \mathrm{M}(J, \mathbb{K})$ by means of the basis (e_j) which restricts to the natural identification of $(\mathrm{GL}_p(H) - 1) \cap B_0(H)$ with $\mathrm{GL}(J, \mathbb{K}) - 1$ if the former endowed with the subspace topology of $\mathrm{B}_0(H)$.

On the other hand Theorem 9 (i) yields that $(\mathrm{GL}_p(H) - 1) \cap \mathrm{B}_0(H)$ and $\mathrm{GL}_p(H) - 1$ are weakly homotopy equivalent – again if the former is endowed with the subspace topology of $\mathrm{B}_0(H)$.

Consequently we get a weak homotopy equivalence between $\mathrm{GL}_p(H)$ and $\mathrm{GL}(J, \mathbb{K})$, the latter group in turn being homotopy equivalent to $\mathrm{GL}(\mathbb{N}, \mathbb{K})$ by Corollary 13.

(ii) The claim follows from the commutative diagram.

$$\begin{array}{ccc} \mathrm{GL}(\mathbb{N}, \mathbb{K}) & \overset{\text{w-}\simeq}{\underset{\text{(i)}}{\hookrightarrow}} & \mathrm{GL}_p(H_s) \\ \text{w-}\simeq \big\downarrow 12 & & \big\downarrow \\ \mathrm{GL}(J, \mathbb{K}) & \overset{\text{w-}\simeq}{\underset{\text{(i)}}{\hookrightarrow}} & \mathrm{GL}_p(H) \end{array}$$

(iii) $\mathrm{B}_p(H) \subseteq \mathrm{B}_q(H)$ is a dense subset; that is, the inclusion $\mathrm{B}_p(H) \hookrightarrow \mathrm{B}_q(H)$ has dense range. Theorem 9 (ii) now shows that the inclusion $\mathrm{GL}_p(H) - 1 \hookrightarrow \mathrm{GL}_q(H) - 1$ is a homotopy equivalence. □

Finally, we want to compute the homotopy groups of the groups

$$\begin{aligned} \mathrm{GL}_p(H, I) &:= \mathrm{GL}_p(H) \cap \mathrm{GL}(H, I) \\ \mathrm{U}_p(H, I) &:= \mathrm{U}_p(H) \cap \mathrm{GL}(H, I) \end{aligned}$$

Observe that we have a *polar decomposition*

$$\mathrm{GL}_p(H, I) \cong \mathrm{U}_p(H, I) \times \mathrm{Herm}_p(H, I)$$

with $\mathrm{Herm}_p(H,I) := \mathrm{Herm}_p(H) \cap \mathrm{Herm}(H,I)$ (inherited from the groups we have intersected).

Corollary 18. *Let H be a infinite-dimensional complex Hilbert space with conjugation I, and $p \in [1,\infty]$. Then the following assertions hold:*

(i) $\pi_k(\mathrm{GL}_p(H,I)) \cong \pi_k(\mathrm{GL}(\mathbb{N},\mathbb{R}))$
(ii) *The inclusion map $\mathrm{GL}_p(H_s, I|_{H_s}) \hookrightarrow \mathrm{GL}_p(H,I)$ is a weak homotopy equivalence for every infinite-dimensional separable I-invariant subspace $H_s \subseteq H$.*
(iii) *The inclusion map $\mathrm{GL}_p(H,I) \hookrightarrow \mathrm{GL}_q(H,I)$ is a homotopy equivalence for $p \leqslant q$.*

Proof. Using polar decomposition we get

$$\mathrm{GL}_p(H,I) \simeq \mathrm{U}_p(H,I) = \mathrm{U}_p(H) \cap \mathrm{U}(H,I)$$

$$\overset{7}{\cong} \mathrm{U}_p(H_\mathbb{R}) \cap \mathrm{U}(H^I_\mathbb{R}) = \mathrm{U}_p(H^I_\mathbb{R}) \simeq \mathrm{GL}_p(H^I_\mathbb{R})$$

Thus all three claims follow from the corresponding assertions of Theorem 17. □

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