

HILBERT-SCHMIDT AND TRACE CLASS OPERATORS

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Let $H \neq 0$ be a Hilbert space. We denote by $B(H)$ and $K(H)$ the algebra of bounded respective compact operators on H and by $B_{\text{fin}}(H)$ the subspace of operator of finite rank. We write \cong if two spaces are isometrically isomorphic. The space of bounded sequences with index set J is denoted by $l^\infty(J)$, its (closed) subspace of zero sequences by $c_0(J)$ and the subspace of sequences with finite support by $c_{\text{fin}}(J)$. The space of (square) summable sequences is written as $l^2(J)$ and $l^1(J)$, respectively.

1. INTRODUCTION

Recall that we have the following hierarchy classic sequence spaces:

$$c_{\text{fin}}(\mathbb{N}) \subseteq l^1(\mathbb{N}) \subseteq l^2(\mathbb{N}) \subseteq c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$$

They are Banach spaces (for $c_{\text{fin}}(\mathbb{N})$) and commutative algebras; $l^2(\mathbb{N})$ even is a Hilbert space.

Similarly, we have the following chain of **operator algebras**:

$$B_{\text{fin}}(H) \subseteq ? \subseteq ? \subseteq K(H) \subseteq B(H)$$

They are Banach spaces (except for $B_{\text{fin}}(H)$) and algebras, although **non-commutative** in general.

The following proposition shows that we can in a sense interpret these operator algebras as the non-commutative analoga of the respective sequence spaces.

1 Proposition. For any orthonormal system (e_n) in H we have an isometric algebra homomorphism

$$\Phi : l^\infty \rightarrow B(H), (a_n) \mapsto x \mapsto \sum_n a_n \langle x, e_n \rangle e_n$$

with $\Phi^{-1}(B_{\text{fin}}(H)) = c_{\text{fin}}$ and $\Phi^{-1}(K(H)) = c_0$.

Proof. We only prove that Φ is well-defined and an isometry:

$$\begin{aligned} \left\| \sum_n a_n \langle x, e_n \rangle e_n \right\|^2 &= \sum_n |a_n|^2 |\langle x, e_n \rangle|^2 \\ &\leq \| (a_n) \|_{l^2}^2 \sum_n |\langle x, e_n \rangle|^2 \leq \| (a_n) \|_{l^2}^2 \|x\|^2 \end{aligned}$$

and $\| \sum_n a_n \langle e_m, e_n \rangle e_n \| = |a_m|$, hence $\| \Phi((a_n)) \| = \| (a_n) \|_{l^2}$. □

It is thus natural to ask the following **questions**: (1) What operator algebras correspond to $l^1(\mathbb{N})$ and $l^2(\mathbb{N})$? (2) Which familiar results from the theory of sequence spaces generalize to the non-commutative case?

2. HILBERT-SCHMIDT OPERATORS

We define the space of **Hilbert-Schmidt operators** as

$$\begin{aligned} B_2(H) &:= \{A \in B(H) : \|A\|_2 < \infty\} \\ \|A\|_2 &:= \sqrt{\sum_{i \in I} \|Ae_i\|^2} \end{aligned}$$

where $(e_i)_{i \in I}$ is an ONB of H . This is a normed space.

An easy calculation shows that this definition does not depend on the choice of basis:

2 Lemma. Let $A \in B(H)$ and let $(e_i)_{i \in I}, (f_j)_{j \in J}$ be two ONBs. Then:

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in J} \|A^* f_j\|^2 \in [0, \infty]$$

Date: Last compiled January 28, 2011; last modified February 8, 2010 or later.
BASED ON [Nee96]

Proof. Suppose the first limit exist. Then by Fubini we have

$$\begin{aligned} \sum_{i \in I} \|Ae_i\|^2 &= \sum_{i \in I} \sum_{j \in J} |\langle Ae_i, f_j \rangle|^2 \\ &= \sum_{j \in J} \sum_{i \in I} |\langle A^* f_j, e_i \rangle|^2 = \sum_{j \in J} \|A^* f_j\|^2, \end{aligned}$$

hence the second limit exists and agrees. \square

The following facts follow easily from the preceding.

3 Proposition. Let $A \in B_2(H)$. Then:

- (i) $\|A^*\|_2 = \|A\|_2$
- (ii) $\|A\| \leq \|A\|_2$
- (iii) $B_2(H)$ is an ‘‘operator ideal’’ in $B(H)$, i.e. $B(H)B_2(H)B(H) \subseteq B_2(H)$

Proof. (i) Lemma 2.

(ii) Let $\epsilon > 0$. Take $e \in H$ such that $\|e\| = 1$ and $\|Ae\| \geq \|A\| - \epsilon$, extend to an ONB (e_i) . Then

$$\|A\|_2^2 = \sum_i \|Ae_i\|^2 \geq \|Ae\|^2 \geq (\|A\| - \epsilon)^2$$

(iii) It is clear that $B_2(H)$ is a left ideal; (i) shows that it is a right ideal. \square

4 Theorem. $(B_2(H), \|\cdot\|_2)$ is a Hilbert space with inner product

$$\langle A, B \rangle_2 := \sum_i \langle B^* A e_i, e_i \rangle = \sum_i \langle A e_i, B e_i \rangle$$

and $B_{\text{fin}}(H)$ is a dense subspace.

Proof. Consider the mapping

$$\Psi : c_{\text{fin}}(I \times I) \rightarrow B_2(H), \delta_{(i,j)} \mapsto \langle \cdot, e_i \rangle e_j$$

From the calculations

$$\begin{aligned} \|\Psi((a_{i,j}))\|^2 &\leq \|\Psi((a_{i,j}))\|_2^2 = \left\| \sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle e_j \right\|_2^2 \\ &= \sum_{i,j} |a_{i,j}|^2 = \|(a_{i,j})\|_{l^2}^2 \end{aligned}$$

we see that Ψ has a continuous extension $l^2(I \times I) \rightarrow B(H)$ which is a surjective isometry onto $B_2(H)$. Thus the latter is also a Hilbert space with dense subspace $\Psi(c_{\text{fin}}(I \times I)) = B_{\text{fin}}(H)$.

The formula for the inner product is easily obtained using the polarization identity. \square

5 Corollary. $B_2(H) \subseteq K(H)$

Proof. Theorem 4 and Proposition 3, (ii). \square

Any Hilbert-Schmidt operator $A \in B_2(H) \subseteq K(H)$ can be written as a series

$$A = \sum_n a_n \langle \cdot, e_n \rangle f_n$$

with $(a_n) \in c_0(\mathbb{N})$ and orthonormal systems $(e_n), (f_n)$. We can easily calculate its norm from any such representation:

6 Proposition.

$$\|A\|_2 = \sqrt{\sum_n |a_n|^2} = \|(a_n)\|_{l^2}$$

Thus a compact operator is a Hilbert-Schmidt operator if and only if its coefficients are in $l^2(\mathbb{N})$.

Finally we will reveal the intimate connection between the Hilbert-Schmidt operators on H and the tensor product of H with its dual.

7 Proposition. The space of Hilbert-Schmidt operator is naturally isometrically isomorphic to the tensor product $H^* \otimes H$ via

$$\Phi : H^* \otimes H \rightarrow B_2(H), \lambda \otimes f \mapsto \lambda f$$

Proof. The mapping Φ is induced by the bilinear map $(\lambda, f) \mapsto \lambda f$, hence well-defined. Choose an ONB (e_i) of H . Then $\langle \cdot, e_i \rangle \otimes e_j$ is an ONB of the tensor product $H^* \otimes H$, and

$$\begin{aligned} \|\Phi(\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle \otimes e_j)\|_2^2 &= \|\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle e_j\|_2^2 \\ &= \sum_{i,j} |a_{i,j}|^2 = \|\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle \otimes e_j\|_2^2 \end{aligned}$$

shows that Φ is an isometry. Thus Φ is also surjective since its range includes the dense set of finite-rank operators. Now apply the bounded inverse theorem. \square

3. TRACE CLASS OPERATORS

We define the space of **trace class** (or **nuclear**) **operators** to be

$$\begin{aligned} B_1(H) &:= \{A \in B_2(H) : \|A\|_1 < \infty\} \\ \|A\|_1 &:= \sup\{|\langle A, B \rangle_2| : B \in B_2(H), \|B\| \leq 1\} \end{aligned}$$

This is a normed space.

Let us first collect some facts about this space.

8 Proposition. Let $A \in B_1(H)$. Then:

- (i) $\|A^*\|_1 = \|A\|_1$
- (ii) $\|A\|_2 \leq \|A\|_1$
- (iii) $B_1(H)$ is an “operator ideal” in $B(H)$, i.e. $B(H)B_1(H)B(H) \subseteq B_1(H)$
- (iv) $B_2(H)B_2(H) \subseteq B_1(H)$

Proof. (i) We have $\langle A, B \rangle_2 = \langle B^*, A^* \rangle_2$ since both sides define inner products inducing the same norm (apply the polarization identity). This in turn implies the claim.

(ii) This follows from $\|A\| \leq \|A\|_2$.

(iii) In view of (i) we only have to show that $B_1(H)$ is a left ideal; this follows readily from $\langle CA, B \rangle_2 = \langle A, C^*B \rangle_2$.

(iv) Let $A, B, C \in B_2(H)$ and $\|B\| \leq 1$. Then

$$\begin{aligned} |\langle CA, B \rangle_2| &= |\langle A, C^*B \rangle_2| \leq \|A\|_2 \|C^*B\|_2 \\ &= \|A\|_2 \|B^*C\|_2 \leq \|A\|_2 \|B^*\| \|C\|_2 \leq \|A\|_2 \|C\|_2, \end{aligned}$$

hence $\|CA\|_1 \leq \|C\|_2 \|A\|_2$. \square

We define the **trace** of a trace class operator $A \in B_1(H)$ to be

$$\text{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle$$

where $(e_i)_{i \in I}$ is an ONB of H . Note that this coincides with the usual definition of the trace if H is finite-dimensional.

The following lemma shows that the definition make sense.

9 Lemma. The series converges absolutely and it is independent from the choice of basis.

Proof. Choose $\lambda_i \in \mathbb{C}$ such that $|\langle Ae_i, e_i \rangle| = \lambda_i \langle Ae_i, e_i \rangle$ and $|\lambda_i| = 1$ ($i \in I$). Then for every finite subset $I_0 \subseteq I$ we have the following estimate:

$$\begin{aligned} \sum_i |\langle Ae_i, e_i \rangle| &= \sum_i \lambda_i \langle Ae_i, e_i \rangle = \sum_i \lambda_i \langle A, \langle \cdot, e_i \rangle e_i \rangle_2 \\ &= \langle A, \sum_i \lambda_i \langle \cdot, e_i \rangle e_i \rangle_2 \leq \|A\|_1 \|\sum_i \lambda_i \langle \cdot, e_i \rangle e_i\| \leq \|A\|_1 \end{aligned}$$

This implies absolute convergence.

If $(f_j)_{j \in J}$ is any other ONB we have

$$\begin{aligned} \sum_i \langle Ae_i, e_i \rangle &= \sum_{i,j} \langle Ae_i, f_i \rangle \langle f_i, e_i \rangle = \sum_{j,i} \langle f_i, e_i \rangle \langle e_i, A^* f_i \rangle \\ &= \sum_j \langle f_j, A^* f_j \rangle = \sum_j \langle A f_j, f_j \rangle, \end{aligned}$$

hence the trace is independent from the particular choice of basis. \square

We now collect some facts about the trace which resemble the finite-dimensional case.

10 Proposition. (i) $\text{tr} \in B_1(H)'$ with $\|\text{tr}\| = 1$

(ii) $\text{tr}(AB) = \text{tr}(BA)$ for $A \in B_1(H)$, $B \in B(H)$ and $A, B \in B_2(H)$, respectively

Proof. (i) By the proof of the preceding lemma we have $\|\text{tr}\| \leq 1$. Equality follows by considering an orthogonal projection.

(ii) If $A \in B_1(H)$ is Hermitian and $B \in B(H)$ we can take an ONB of eigenvectors (e_i) with $Ae_i =: \lambda_i e_i$ for real eigenvalues $\lambda_i \in \mathbb{R}$. Then

$$\begin{aligned} \text{tr}(AB) &= \sum_i \langle AB e_i, e_i \rangle = \sum_i \langle B e_i, A e_i \rangle \\ &= \sum_i \langle B A e_i, e_i \rangle = \text{tr}(BA) \end{aligned}$$

If $A \in B_1(H)$ is a general trace class operator we can still write it as a sum $A = B + iC$ with Hermitian $B, C \in B_1(H)$. The claim then follows from the complex bilinearity of $(A, B) \mapsto \text{tr}(AB)$ and $(A, B) \mapsto \text{tr}(BA)$.

For $A, B \in B_2(H)$ the claim follows from

$$\begin{aligned} \text{tr}(AB) &= \sum_i \langle AB e_i, e_i \rangle = \langle B, A^* \rangle \\ &= \langle A, B^* \rangle = \sum_i \langle B A e_i, e_i \rangle = \text{tr}(BA) \end{aligned} \quad \square$$

Let us write a trace class operator $A \in B_1(H)$ as a series

$$A = \sum_n a_n \langle \cdot, e_n \rangle f_n$$

with $(a_n) \in c_0(\mathbb{N})$ and orthonormal systems $(e_n), (f_n)$. Again it is easy to calculate its norm and trace from this representation:

11 Proposition.

$$\begin{aligned} \|A\|_1 &= \sum_n |a_n| = \|(a_n)\|_{l^1} \\ \text{tr}(A) &= \sum_n a_n \langle f_n, e_n \rangle \end{aligned}$$

Thus a compact operator is a trace class operator if and only if its coefficients are in $l^1(\mathbb{N})$.

Proof. We only show the first equality; the second one is immediate from the definition of tr .

(\leq) For any $B \in B_2(H)$ with $\|B\| \leq 1$ we have

$$\begin{aligned} |\langle A, B \rangle_2| &\leq \sum_n |a_n| |\langle \langle \cdot, e_n \rangle f_n, B \rangle_2| \\ &\leq \sum_n |a_n| |\langle f_n, B e_n \rangle| \leq \sum_n |a_n|, \end{aligned}$$

hence $\|A\|_1 \leq \sum_n |a_n|$.

(\geq) Choose $b_n \in \mathbb{C}$ such that $|a_n| = a_n b_n$ and $|b_n| = 1$ ($n \in \mathbb{N}$) and define

$$B_N := \sum_{n=1}^N b_n \langle \cdot, e_n \rangle f_n$$

Clearly $B_N \in B_2(H)$ and $\|B_N\| \leq 1$. Hence

$$\begin{aligned} \|A\|_1 &\geq |\langle A, B_N \rangle_2| = \left| \sum_n \langle Ae_n, B_N e_n \rangle \right| \\ &= \left| \sum_{n=1}^N a_n b_n \right| = \sum_{n=1}^N |a_n| \uparrow \sum_n |a_n| \end{aligned} \quad \square$$

It follows that we can approximate any trace class operator using finite rank operators:

12 Corollary. B_{fin} is a dense subspace of $(B_1(H), \|\cdot\|_1)$.

Proof. We have

$$\|A - \sum_{n=1}^N a_n \langle \cdot, e_n \rangle f_n\|_1 \stackrel{11}{=} \sum_{n=N}^{\infty} |a_n| \rightarrow 0$$

as $N \rightarrow \infty$. □

We can also deduce that every trace class operator is the product of two Hilbert-Schmidt operators:

13 Proposition. $B_2(H)B_2(H) = B_1(H)$

Proof. (\subseteq) was proved in Proposition 8, (iv).

(\supseteq) Define

$$\begin{aligned} B &= \sum_n \sqrt{a_n} \langle \cdot, e_n \rangle f_n \\ C &= \sum_n \sqrt{a_n} \langle \cdot, e_n \rangle e_n \end{aligned}$$

Then B and C are Hilbert-Schmidt operators, and $A = BC$. □

Note that

$$B_1(H) \times B(H) \rightarrow \mathbb{C}, (A, B) \mapsto \text{tr}(AB)$$

is a continuous pairing since we have

$$|\text{tr}(AB)| \leq \|AB\|_1 \leq \|A\|_1 \|B\|$$

This pairing induces the following two isometric isomorphisms.

14 Theorem. $B_1(H) \cong K(H)'$ and $B_1(H)' \cong B(H)$

Proof. (1) We show that

$$B_1(H) \rightarrow K(H)', B \mapsto \text{tr}(\cdot B)$$

is an isometric isomorphism.

Linearity is obvious. It is almost by definition of $\|\cdot\|_1$ that the mapping is an isometry. Hence it remains to show surjectivity. Let $\varphi \in K(H)'$. Then for all $A \in B_2(H)$ we have

$$|\varphi(A)| \leq \|\varphi\| \|A\| \leq \|\varphi\| \|A\|_2,$$

hence $\varphi|_{B_2(H)} \in B_2(H)'$. Take the unique $B \in B_2(H)$ such that

$$\varphi|_{B_2(H)} = \langle \cdot, B \rangle_2 = \langle B^*, \cdot \rangle_2 = \text{tr}(\cdot B^*)$$

From this we see that the continuity of φ implies that B^* is of trace class, and density of $B_2(H) \subseteq K(H)$ shows that B^* is a preimage of φ .

(2) We show that

$$B(H) \rightarrow B_1(H)', B \mapsto \text{tr}(\cdot B)$$

is an isometric isomorphism.

Again, it is obvious that the mapping is a linear isometry. We show surjectivity. Let $\varphi \in B_1(H)'$. Since for $e, f \in H$

$$\|\langle \cdot, e \rangle f\|_1 = \|e\| \|f\|$$

we see that the mapping $f \mapsto \varphi(\langle \cdot, e \rangle f)$ is in H' . Hence there is a unique $\varphi_e \in H$ such that

$$\langle f, \varphi_e \rangle = \varphi(\langle \cdot, e \rangle f) \quad \forall f \in H$$

TABLE 1. Comparison of sequence and operator spaces

Sequence spaces	Operator spaces
$c_{\text{fin}}(\mathbb{N})$ dense in $l^1(\mathbb{N})$,	$B_{\text{fin}}(H)$ dense in $B_1(H)$,
$l^2(\mathbb{N})$ and $c_0(\mathbb{N})$	$B_2(H)$ and $K(H)$
$l^1(\mathbb{N}) = l^2(\mathbb{N})l^2(\mathbb{N})$	$B_1(H) = B_2(H)B_2(H)$
$(a_n) \mapsto \sum_n a_n \in l^1(\mathbb{N})'$	$\text{tr} \in B_1(H)'$
$c_0(\mathbb{N})' \cong l_1(\mathbb{N})$	$K(H)' \cong B_1(H)$
$l_1(\mathbb{N})' \cong l^\infty(\mathbb{N})$	$B_1(H)' \cong B(H)$

and $\|\varphi_e\| \leq \|\varphi\| \|e\|$. Thus $B : H \rightarrow H, e \mapsto \varphi_e$ defines a bounded operator. And the calculation

$$\varphi(\langle \cdot, e \rangle f) = \langle f, Be \rangle = \langle \langle \cdot, e \rangle f, B \rangle_2 = \text{tr}(\langle \cdot, e \rangle f B^*)$$

together with density of $B_{\text{fin}}(H) \subseteq B_1(H)$ shows that B^* is a preimage of φ . \square

15 Corollary. $(B_1(H), \|\cdot\|_1)$ is a Banach space.

4. SUMMARY

Propositions 6 and 11 show that the algebras of Hilbert-Schmidt and trace class operators are the natural non-commutative analoga of $l^1(\mathbb{N})$ and $l^2(\mathbb{N})$, respectively. That is, we have the following chains which correspond in the sense of Proposition 1:

$$\begin{aligned} c_{\text{fin}}(\mathbb{N}) &\subseteq l^1(\mathbb{N}) \subseteq l^2(\mathbb{N}) \subseteq c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N}) \\ B_{\text{fin}}(H) &\subseteq B_1(H) \subseteq B_2(H) \subseteq K(H) \subseteq B(H) \end{aligned}$$

In table 1 we have summarized some familiar facts about sequence spaces together with their non-commutative counterparts (which we have proved in the preceding).

REFERENCES

- [Nee96] Karl-Hermann Neeb. Skript zur Vorlesung Spektral- und Darstellungstheorie. <http://www.mathematik.tu-darmstadt.de/fbereiche/AlgGeoFA/staff/neeb/skripten/Spektraltheorie-SS96.pdf>, 1996.