HILBERT- SCHMIDT AND TRACE CLASS OPERATORS

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Let $H \neq 0$ be a Hilbert space. We denote by $B(H)$ and $K(H)$ the algebra of bounded respective compact operators on $H$ and by $B_{\text{fin}}(H)$ the subspace of operator of finite rank. We write $\cong$ if two spaces are isometrically isomorphic. The space of bounded sequences with index set $J$ is denoted by $l^\infty(J)$, its (closed) subspace of zero sequences by $c_0(J)$ and the subspace of sequences with finite support by $c_{\text{fin}}(J)$. The space of (square) summable sequences is written as $l^2(J)$ and $l^1(J)$, respectively.

1. INTRODUCTION

Recall that we have the following hierarchy classic sequence spaces:

$$c_{\text{fin}}(\mathbb{N}) \subseteq l^1(\mathbb{N}) \subseteq l^2(\mathbb{N}) \subseteq c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$$

They are Banach spaces (for $c_{\text{fin}}(\mathbb{N})$) and commutative algebras; $l^2(\mathbb{N})$ even is a Hilbert space.

Similarly, we have the following chain of operator algebras:

$$B_{\text{fin}}(H) \subseteq? \subseteq? \subseteq K(H) \subseteq B(H)$$

They are Banach spaces (except for $B_{\text{fin}}(H)$) and algebras, although non-commutative in general.

The following proposition shows that we can in a sense interprete these operator algebras as the non-commutative analoga of the respective sequence spaces.

**Proposition 1.** For any orthonormal system $(e_n)$ in $H$ we have an isometric algebra homomorphism

$$\Phi : l^\infty \rightarrow B(H), (a_n) \mapsto x \mapsto \sum_n a_n \langle x, e_n \rangle e_n$$

with $\Phi^{-1}(B_{\text{fin}}(H)) = c_{\text{fin}}$ and $\Phi^{-1}(K(H)) = c_0$.

**Proof.** We only prove that $\Phi$ is well-defined and an isometry:

$$\|\sum_n a_n \langle x, e_n \rangle e_n\|^2 = \sum_n |a_n|^2 \langle x, e_n \rangle^2$$

$$\leq \|\langle a_n \rangle\|^2 \sum_n \|\langle x, e_n \rangle\|^2 = \|\langle a_n \rangle\|^2 \|x\|^2$$

and $\|\sum_n a_n \langle e_m, e_n \rangle e_n\| = |a_n|$, hence $\|\Phi(a_n)\| = \|\langle a_n \rangle\|_\infty$.

It is thus natural to ask the following **questions**: (1) What operator algebras correspond to $l^1(\mathbb{N})$ and $l^2(\mathbb{N})$? (2) Which familiar results from the theory of sequence spaces generalize to the non-commutative case?

2. HILBERT-SCHMIDT OPERATORS

We define the space of **Hilbert-Schmidt operators** as

$$B_2(H) := \{ A \in B(H) : \|A\|_2 < \infty \}$$

$$\|A\|_2 := \sqrt{\sum_{i \in I} \|Ae_i\|^2}$$

where $(e_i)_{i \in I}$ is an ONB of $H$. This is a normed space.

An easy calculation shows that this definition does not depend on the choice of basis:

**Lemma 2.** Let $A \in B(H)$ and let $(e_i)_{i \in I}$, $(f_j)_{j \in J}$ be two ONBs. Then:

$$\sum_{i \in I} \|A e_i\|^2 = \sum_{j \in J} \|A^* f_j\|^2 \in [0, \infty]$$
Proof. Suppose the first limit exist. Then by Fubini we have
\[
\sum_{i \in I} ||Ae_i||^2 = \sum_{i \in I} \sum_{j \in J} |(Ae_i, f_j)|^2
\]
\[
= \sum_{j \in J} \sum_{i \in I} |(A^* f_j, e_i)|^2 = \sum_{j \in J} ||A^* f_j||^2,
\]
hence the second limit exists and agrees. \(\square\)

The following facts follow easily from the preceding.

**Proposition 3.** Let \( A \in B_2(H) \). Then:

(i) \( ||A^*||_2 = ||A||_2 \)

(ii) \( ||A|| \le ||A||_2 \)

(iii) \( B_2(H) \) is an “operator ideal” in \( B(H) \), i.e. \( B(H)B_2(H)B(H) \subseteq B_2(H) \)

Proof. (i) Lemma 2.

(ii) Let \( \epsilon > 0 \). Take \( e \in H \) such that \( ||e|| = 1 \) and \( ||Ae|| \ge ||A|| - \epsilon \), extend to an ONB \( (e_i) \). Then
\[
||A||_2^2 = \sum_i ||Ae_i||^2 \ge ||Ae||^2 \ge (||A|| - \epsilon)^2
\]

(iii) It is clear that \( B_2(H) \) is a left ideal; (i) shows that it is a right ideal. \( \square \)

**Theorem 4.** \( (B_2(H), || \cdot ||_2) \) is a Hilbert space with inner product
\[
\langle A, B \rangle_2 := \sum_i \langle B^* A e_i, e_i \rangle = \sum_i \langle A e_i, B e_i \rangle
\]
and \( B_{fin}(H) \) is a dense subspace.

Proof. Consider the mapping
\[
\Psi : c_{fin}(I \times I) \to B_2(H), \delta_{(i,j)} \mapsto \langle \cdot, e_i \rangle e_j
\]
From the calculations
\[
||\Psi((a_{i,j}))||^2 \le ||\Psi((a_{i,j}))||_2^2 = ||\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle e_j ||_2^2
\]
\[
= \sum_{i,j} ||a_{i,j}||^2 = ||(a_{i,j})||_2^2
\]
we see that \( \Psi \) has a continuous extension \( \ell^2(I \times I) \to B(H) \) which is a surjective isometry onto \( B_2(H) \). Thus the latter is also a Hilbert space with dense subspace \( \Psi(c_{fin}(I \times I)) = B_{fin}(H) \).

The formula for the inner product is easily obtained using the polarization identity. \( \square \)

**Corollary 5.** \( B_2(H) \subseteq K(H) \)

Proof. Theorem 4 and Proposition 3, (ii).

Any Hilbert-Schmidt operator \( A \in B_2(H) \subseteq K(H) \) can be written as a series
\[
A = \sum_n a_n \langle \cdot, e_n \rangle f_n
\]
with \( (a_n) \in c_0(\mathbb{N}) \) and orthonormal systems \( (e_n), (f_n) \). We can easily calculate its norm from any such representation:

**Proposition 6.**
\[
||A||_2 = \left( \sum_n ||a_n||^2 \right)^{1/2} = ||(a_n)||_2^{1/2}
\]
Thus a compact operator is a Hilbert-Schmidt operator if and only if its coefficients are in \( \ell^2(\mathbb{N}) \).

Finally we will reveal the intimate connection between the Hilbert-Schmidt operators on \( H \) and the tensor product of \( H \) with its dual.
Proposition 7. The space of Hilbert-Schmidt operator is naturally isometrically isomorphic to the tensor product $H^* \otimes H$ via

$$\Phi : H^* \otimes H \to B_2(H), \lambda \otimes f \mapsto \lambda f$$

Proof. The mapping $\Phi$ is induced by the bilinear map $(\lambda, f) \mapsto \lambda f$, hence well-defined. Choose an ONB $(e_i)$ of $H$. Then $\langle \cdot, e_i \rangle \otimes e_j$ is an ONB of the tensor product $H^* \otimes H$, and

$$||\Phi(\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle \otimes e_j)||^2 = \sum_{i,j} |a_{i,j}|^2 = \sum_{i,j} ||a_{i,j} \langle \cdot, e_i \rangle \otimes e_j||^2$$

shows that $\Phi$ is an isometry. Thus $\Phi$ is also surjective since its range includes the dense set of finite-rank operators. Now apply the bounded inverse theorem. \hfill \square

3. Trace class operators

We define the space of trace class (or nuclear) operators to be

$$B_1(H) := \{ A \in B_2(H) : ||A||_1 < \infty \}$$

$$||A||_1 := \sup \{ ||(A, B)_2| : B \in B_2(H), ||B|| \leq 1 \}$$

This is a normed space.

Let us first collect some facts about this space.

Proposition 8. Let $A \in B_1(H)$. Then:

(i) $||A^*||_1 = ||A||_1$

(ii) $||A||_2 \leq ||A||_1$

(iii) $B_1(H)$ is an “operator ideal” in $B(H)$, i.e. $B(H)B_1(H)B(H) \subseteq B_1(H)$

(iv) $B_2(H)B_2(H) \subseteq B_1(H)$

Proof. (i) We have $\langle A, B \rangle_2 = \langle B^*, A^* \rangle_2$ since both sides define inner products inducing the same norm (apply the polarization identity). This in turn implies the claim.

(ii) This follows from $||A|| \leq ||A||_2$.

(iii) In view of (i) we only have to show that $B_1(H)$ is a left ideal; this follows readily from $\langle CA, B \rangle_2 = \langle A, C^*B \rangle_2$.

(iv) Let $A, B, C \in B_2(H)$ and $||B|| \leq 1$. Then

$$||\langle CA, B \rangle_2| = ||\langle A, C^*B \rangle_2| \leq ||A||_2||C^*B||_2$$

$$\leq ||A||_2||B^*C||_2 \leq ||A||_2||B^*||||C||_2 \leq ||A||_2||C||_2,$$

hence $||CA||_1 \leq ||C||_2||A||_2$. \hfill \square

We define the trace of a trace class operator $A \in B_1(H)$ to be

$$\text{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle$$

where $(e_i)_{i \in I}$ is an ONB of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional.

The following lemma shows that the definition make sense.

Lemma 9. The series converges absolutely and it is independent from the choice of basis.

Proof. Choose $\lambda_i \in \mathbb{C}$ such that $||\langle Ae_i, e_i \rangle| = \lambda_i \langle Ae_i, e_i \rangle$ and $|\lambda_i| = 1$ ($i \in I$). Then for every finite subset $I_0 \subseteq I$ we have the following estimate:

$$\sum_i |\langle Ae_i, e_i \rangle| = \sum_i \lambda_i \langle Ae_i, e_i \rangle = \sum_i \lambda_i \langle A, \langle \cdot, e_i \rangle e_i \rangle_2$$

$$= \langle A, \sum_i \lambda_i \langle \cdot, e_i \rangle e_i \rangle_2 \leq ||A||_1 \left( \sum_i ||\langle \cdot, e_i \rangle e_i|| \right) \leq ||A||_1$$

This implies absolute convergence.
If \((f_j)_{j \in J}\) is any other ONB we have
\[
\sum_i \langle Ae_i, e_i \rangle = \sum_{i,j} \langle Ae_i, f_j \rangle \langle f_j, e_i \rangle = \sum_{j,i} \langle f_j, e_i \rangle \langle e_i, A^* f_j \rangle = \sum_j \langle f_j, A^* f_j \rangle = \sum_j \langle A f_j, f_j \rangle,
\]
hence the trace is independent from the particular choice of basis.

We now collect some facts about the trace which resemble the finite-dimensional case.

**Proposition 10.**
\(\text{(i) } \text{tr} \in B_1(H)' \text{ with } ||\text{tr}|| = 1\)
\(\text{(ii) } \text{tr}(AB) = \text{tr}(BA) \text{ for } A \in B_1(H), B \in B(H) \text{ and } A, B \in B_2(H), \text{ respectively}\)

**Proof.** (i) By the proof of the preceding lemma we have ||tr|| ≤ 1. Equality follows by considering an orthogonal projection.

(ii) If \(A \in B_1(H)\) is Hermitian and \(B \in B(H)\) we can take an ONB of eigenvectors \((e_i)\) with \(A e_i =: \lambda_i e_i\) for real eigenvalues \(\lambda_i \in \mathbb{R}\). Then
\[
\text{tr}(AB) = \sum_i \langle A B e_i, e_i \rangle = \sum_i \langle B e_i, A e_i \rangle = \sum_i \langle B A e_i, e_i \rangle = \text{tr}(BA)
\]

If \(A \in B_1(H)\) is a general trace class operator we can still write it as a sum \(A = B + iC\) with Hermitian \(B, C \in B_1(H)\). The claim then follows from the complex bilinearity of \((A, B) \mapsto \text{tr}(AB)\) and \((A, B) \mapsto \text{tr}(BA)\).

For \(A, B \in B_2(H)\) the claim follows from
\[
\text{tr}(AB) = \sum_i \langle A B e_i, e_i \rangle = \langle B, A^* \rangle = \langle A, B^* \rangle = \sum_i \langle B A e_i, e_i \rangle = \text{tr}(BA)
\]

Let us write a trace class operator \(A \in B_1(H)\) as a series
\[
A = \sum_n a_n \langle \cdot, e_n \rangle f_n
\]
with \((a_n) \in c_0(\mathbb{N})\) and orthonormal systems \((e_n)\), \((f_n)\). Again it is easy to calculate its norm and trace from this representation:

**Proposition 11.**
\[
||A||_1 = \sum_n |a_n| = ||(a_n)||_1
\]
\[
\text{tr}(A) = \sum_n a_n (f_n e_n)
\]
Thus a compact operator is a trace class operator if and only if its coefficients are in \(l^1(\mathbb{N})\).

**Proof.** We only show the first equality; the second one is immediate from the definition of tr.

(≤) For any \(B \in B_2(H)\) with \(||B|| \leq 1\) we have
\[
|\langle A, B \rangle_2| \leq \sum_n |a_n| ||\langle \cdot, e_n \rangle f_n, B \rangle_2| \leq \sum_n |a_n| ||f_n, B e_n|| \leq \sum_n |a_n|,
\]
hence \(||A||_1 \leq \sum_n |a_n|\).

(≥) Choose \(b_n \in \mathbb{C}\) such that \(|a_n| = a_n b_n\) and \(|b_n| = 1\) \((n \in \mathbb{N})\) and define
\[
B_N := \sum_{n=1}^N b_n \langle \cdot, e_n \rangle f_n
\]
Clearly $B_N \in B_2(H)$ and $\|B_N\| \leq 1$. Hence

$$\|A\|_1 \geq \|A, B_N\|_2 = \sum_{n=1}^{N} \langle A e_n, B_N e_n \rangle \|$$

$$= \sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} |a_n| \uparrow \sum_{n=1}^{N} |a_n|$$

It follows that we can approximate any trace class operator using finite rank operators:

**Corollary 12.** $B_{\text{fin}}$ is a dense subspace of $(B_1(H), \| \cdot \|_1)$.

**Proof.** We have

$$\|A - \sum_{n=1}^{N} a_n(\cdot, e_n) f_n\|_1 \leq \sum_{n=N}^{\infty} |a_n| \to 0$$

as $N \to \infty$.

We can also deduce that every trace class operator is the product of two Hilbert-Schmidt operators:

**Proposition 13.** $B_2(H)B_2(H) = B_1(H)$

**Proof.** ($\subseteq$) was proved in Proposition 8, (iv).

($\supseteq$) Define

$$B = \sum_n \sqrt{a_n}(\cdot, e_n) f_n$$

$$C = \sum_n \sqrt{a_n}(\cdot, e_n) e_n$$

Then $B$ and $C$ are Hilbert-Schmidt operators, and $A = BC$.

Note that

$$B_1(H) \times B(H) \to \mathbb{C}, \ (A, B) \mapsto \text{tr}(AB)$$

is a continuous pairing since we have

$$|\text{tr}(AB)| \leq \|AB\|_1 \leq \|A\|_1 \|B\|$$

This pairing induces the following two isometric isomorphisms.

**Theorem 14.** $B_1(H) \cong K(H)'$ and $B_1(H)' \cong B(H)$

**Proof.** (1) We show that

$$B_1(H) \to K(H)', B \mapsto \text{tr}(\cdot B)$$

is an isometric isomorphism.

Linearity is obvious. It is almost by definition of $\| \cdot \|_1$ that the mapping is an isometry. Hence it remains to show surjectivity. Let $\varphi \in K(H)'$. Then for all $A \in B_2(H)$ we have

$$|\varphi(A)| \leq \|\varphi\|\|A\| \leq \|\varphi\|\|A\|_2,$$

hence $\varphi|_{B_2(H)} \in B_2(H)'$. Take the unique $B \in B_2(H)$ such that

$$\varphi|_{B_2(H)} = \langle \cdot, B \rangle_2 = \langle B^*, \cdot \rangle_2 = \text{tr}(\cdot B^*)$$

From this we see that the continuity of $\varphi$ implies that $B^*$ is of trace class, and density of $B_2(H) \subseteq K(H)$ shows that $B^*$ is a preimage of $\varphi$.

(2) We show that

$$B(H) \to B_1(H)', B \mapsto \text{tr}(\cdot B)$$

is an isometric isomorphism.

Again, it is obvious that the mapping is a linear isometry. We show surjectivity. Let $\varphi \in B_1(H)'$. Since for $e, f \in H$

$$\|\langle \cdot, e \rangle f_1 = \|e\|\|f\|$$
we see that the mapping $f \mapsto \varphi(\langle \cdot, e \rangle f)$ is in $H'$. Hence there is a unique $\varphi_e \in H$ such that
\[
\langle f, \varphi_e \rangle = \varphi(\langle \cdot, e \rangle f) \quad \forall f \in H
\]
and $||\varphi_e|| \leq ||\varphi|| ||e||$. Thus $B : H \to H, e \mapsto \varphi_e$ defines a bounded operator. And the calculation
\[
\varphi(\langle \cdot, e \rangle f) = \langle f, B e \rangle = \langle \langle \cdot, e \rangle f, B \rangle_2 = \text{tr}(\langle \cdot, e \rangle f B^*)
\]
together with density of $B_{\text{fin}}(H) \subseteq B_1(H)$ shows that $B^*$ is a preimage of $\varphi$. \hfill $\square$

**Corollary 15.** $(B_1(H), ||\cdot||_1)$ is a Banach space.

4. Summary

Propositions 6 and 11 show that the algebras of Hilbert-Schmidt and trace class operators are the natural non-commutative analogs of $l^1(\mathbb{N})$ and $l^2(\mathbb{N})$, respectively. That is, we have the following chains which correspond in the sense of Proposition 1:

- $c_{\text{fin}}(\mathbb{N}) \subseteq l^1(\mathbb{N}) \subseteq l^2(\mathbb{N}) \subseteq c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$
- $B_{\text{fin}}(H) \subseteq B_1(H) \subseteq B_2(H) \subseteq K(H) \subseteq B(H)$

In table 1 we have summarized some familiar facts about sequence spaces together with their non-commutative counterparts (which we have proved in the preceding).

**References**