STIEFEL-WHITNEY CLASSES
I. AXIOMS AND CONSEQUENCES

MICHAEL WALTER

Abstract. After a brief review of cohomology theory we define the Stiefel-Whitney classes associated to a vector bundle and prove some consequences from their axioms. We proceed to compute the Stiefel-Whitney classes of projective space and apply the result to show non-existence of real division algebras in most dimensions.

Notation. We denote by \( \epsilon^n \) the trivial \( n \)-dimensional vector bundle over a given space. Isomorphism in the respective category is denoted by \( \cong \) (e.g. homeomorphism for topological spaces, isomorphism of Abelian groups, equivalence of bundles over a fixed base space).

Acknowledgements. Sections 2 and 3 are based on the presentation in [MS74, §4].

1. Cohomology Theory

Since the Stiefel-Whitney classes of a vector bundle are invariants which live in the cohomology groups of the base space we shall give a brief review of cohomology theory (cf. [Hat02], [Lü05] or [May99] for more detailed accounts).

Axioms. A cohomology theory with coefficients in an \( R \)-module \( M \) is a contravariant functor

\[ H^*(\cdot; M) : \text{Topological pairs} \to \mathbb{Z}\text{-graded Abelian groups} \]

together with natural transformations

\[ \partial : H^k(A; M) \to H^{k+1}(X, A; M) \]

satisfying the Eilenberg-Steenrod axioms\(^\text{1}\):

(C1) homotopy-invariance: any two homotopic maps induce the same morphism

(C2) long exact sequence: any two homotopic maps induce a long exact sequence of the form

\[
\ldots \to H^k(X, A; M) \xrightarrow{\text{id}^*} H^k(X; M) \xrightarrow{\text{incl}^*} H^k(A; M) \xrightarrow{\partial} H^{k+1}(X, A; M) \to \ldots
\]

(C3) excision: given subspaces \( \text{cl}(Z) \subseteq \text{int}(A) \subseteq X \) we have induced isomorphisms

\[ H^k(X, A; M) \xrightarrow{\text{incl}^*} H^k(X \setminus Z, A \setminus Z; M) \]

\(^{1}\)We write \( X \) for a pair \( (X, \emptyset) \) and we denote by \( f^* \) the morphism \( H^k(f) \) induced by a map \( f \).
(C4) product axiom: given a family of topological pairs \((X_i, A_i)\) we have induced isomorphisms

\[ H^k(\prod_i (X_i, A_i); M) \xrightarrow{\prod_i \text{incl}^*} \prod_i H^k(X_i, A_i; M) \]

An ordinary cohomology theory also satisfies the following axiom:

(C5) dimension axiom:

\[ H^k(\{\ast\}; M) = \begin{cases} M, & k = 0 \\ 0, & k \neq 0 \end{cases} \]

1 Theorem. There exists a cohomology theory for an arbitrary coefficient module, called singular cohomology.

Reduced Cohomology. For calculations it is often a nuisance that the cohomology groups of a point are trivial. This motivates the definition of the reduced cohomology groups

\[ \tilde{H}^k(X; M) := H^k(X, \{\ast\}; M) \]

2 Proposition.

\[ H^k(X; M) \cong \tilde{H}^k(X; M) \oplus H^k(\{\ast\}; M) \]

In particular, points (and hence contractible spaces) have trivial reduced cohomology.

Proof. By the long exact sequence (2)

\[ \ldots \longrightarrow \tilde{H}^k(X; M) \xrightarrow{\text{id}^*} H^k(X; M) \xrightarrow{\text{incl}^*} H^k(\{\ast\}; M) \xrightarrow{\partial} \tilde{H}^{k+1}(X; M) \longrightarrow \ldots \]

Now, every space retracts to a point, i.e. \(\text{const} \circ \text{incl} = \text{id}_{\{\ast\}}\). Hence \(\text{incl}^* \circ \text{const}^*_\ast = \text{id}_{H^k}\) and we have a split exact sequence

\[ 0 \longrightarrow \tilde{H}^k(X; M) \xrightarrow{\text{id}^*} H^k(X; M) \xrightarrow{\text{incl}^*} H^k(\{\ast\}; M) \longrightarrow 0 \]

\[ \text{□} \]

There is also a long exact sequence for reduced cohomology:

3 Theorem. Any pair \((X, A)\) induces a long exact sequence of the form

\[ \ldots \longrightarrow H^k(X, A; M) \longrightarrow \tilde{H}^k(X; M) \longrightarrow \tilde{H}^k(A; M) \longrightarrow H^{k+1}(X, A; M) \longrightarrow \ldots \]

(Observe that the base point \(\ast\) has to be chosen in \(A\).)

Proof. By functoriality, naturality and the long exact sequence (2) we have the following diagram with exact rows and columns:

\[ \begin{array}{cccccccc}
\ldots & H^k(X, A; M) & \longrightarrow & \tilde{H}^k(X; M) & \longrightarrow & \tilde{H}^k(A; M) & \longrightarrow & H^{k+1}(X, A; M) & \longrightarrow & \ldots \\
\downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id}^* & \downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id} \\
\ldots & H^k(X, A; M) & \longrightarrow & H^k(X; M) & \longrightarrow & H^k(A; M) & \longrightarrow & \tilde{H}^{k+1}(X, A; M) & \longrightarrow & \ldots \\
\downarrow \text{incl}^* = 0 & \downarrow \text{incl}^* & \downarrow \text{incl}^* & \downarrow \text{incl}^* & \downarrow \text{incl}^* & \downarrow \text{incl}^* & \downarrow \text{incl}^* & \downarrow \text{incl}^* = 0 & \downarrow \text{incl}^* = 0 & \downarrow \text{incl}^* = 0 \\
\ldots & H^k(\{\ast\}, \{\ast\}; M) & \longrightarrow & H^k(\{\ast\}; M) & \longrightarrow & H^k(\{\ast\}; M) & \longrightarrow & \tilde{H}^{k+1}(\{\ast\}, \{\ast\}; M) & \longrightarrow & \ldots \\
\end{array} \]

The existence of the dotted maps in the first row follows from diagram chasing. \(\text{□} \)
Spheres. Long exact sequences are the main tool in computing the cohomology of spaces.

Example 4. The reduced cohomology groups of the spheres are given by

\[ \tilde{H}^k(S^n; M) = \begin{cases} M, & n = k \\ 0, & \text{otherwise} \end{cases} \]

Proof. (1) The assertion for \( S^0 = \{ \pm 1 \} \) is immediate from the product and dimension axioms \((C4)\) and \((C5)\).

(2) On the other hand it follows from the long exact reduced cohomology sequence for the pair \((D^{n+1}, S^n)\) that

\[ \tilde{H}^k(S^n; M) \cong H^{k+1}(D^{n+1}, S^n; M) \]

(3) Now consider the “wrap-around map” \( f : (D^{n+1}, S^n) \to (S^{n+1}, \{\ast\}) \) which is a homeomorphism away from the boundary (it is the inverse of stereographic projection from \( \ast \)). We define subspaces

\[ S^n \subseteq (D^{n+1} \setminus \frac{3}{4} D^{n+1}) \subseteq \frac{1}{2} D^{n+1} \subseteq D^{n+1}, \]

\[ \ast \in f(Z) \subseteq f(A) \subseteq S^{n+1} \]

and get a commutative diagram

\[
\begin{array}{ccc}
(D^{n+1}, S^n) & \xrightarrow{f} & (S^{n+1}, \{\ast\}) \\
\downarrow \cong & & \downarrow \cong \\
(D^{n+1}, A) & \xrightarrow{f} & (S^{n+1}, f(A)) \\
\downarrow \cong & & \downarrow \cong \\
(D^{n+1} \setminus Z, A \setminus Z) & \xrightarrow{f} & (S^{n+1} \setminus f(Z), f(A) \setminus f(Z))
\end{array}
\]

The upper inclusions are homotopy equivalences, the lower inclusions satisfy the hypotheses of excision and the bottom restriction is a homeomorphism. Hence they induce isomorphism of the cohomology groups and it follows that \( f \) induces the first isomorphism in

\[ \tilde{H}^{k+1}(S^{n+1}; M) \cong H^{k+1}(D^{n+1}, S^n; M) \cong \tilde{H}^k(S^n; M) \]

The claim now follows by induction. \(\square\)

Multiplicative Structure. A multiplicative structure on a cohomology theory \( H^\ast(\_; M) \) with coefficients in an \( R \)-module \( M \) is given by a family of \( R \)-linear maps

\[ \cup : H^k(X, A; M) \times H^l(X, B; M) \to H^{k+l}(X, A \cup B) \]

(callled the cup product) together with a unit \( 1_X \in H^0(X; M) \) satisfying the following axioms:

\( \text{(M1) associativity} \)

\( \text{(M2) graded commutativity:} \)

\[ x \cup y = (-1)^{kl} y \cup x \quad (\forall x \in H^k(X, A; M), y \in H^l(X, B; M)) \]

\( \text{(M3) naturality} \)
(M4) compatibility with the boundary map: if \((X, A)\) is any pair, then 
\[
\partial(a) \cup x = \partial(a \cup \text{incl}^*(x)) \quad (\forall a \in H^k(A; M), x \in H^1(X; M))
\]

We will usually omit the \(\cup\) symbol from products. The product turns the groups

\[
H^*(X, A; M) := \bigoplus_{k=0}^\infty H^k(X, A; M) \quad \text{and} \quad H^{**}(X, A; M) := \prod_{k=0}^\infty H^k(X, A; M)
\]

into unital graded commutative rings, called the cohomology rings.

5 Lemma. Elements of the form \(1_X + x\) where the zeroth coefficient of \(x\) vanishes are invertible in \(H^{**}(X, A; M)\).

6 Theorem. Singular cohomology admits a multiplicative structure if \(M = R\) is a commutative unital ring.

**Projective Space.** The following theorem describes the structure of the singular cohomology ring of projective space with \(\mathbb{Z}_2\)-coefficients. Note that this is a commutative ring since \(-1 = 1\) in \(\mathbb{Z}_2\).

7 Theorem. The cohomology groups of projective space with \(\mathbb{Z}_2\)-coefficients are given by

\[
H^k(P^n; \mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2 & , k \leq n \\
0 & , \text{otherwise}
\end{cases}
\]

and the higher cohomology groups are generated by the respective powers of the generator \(0 \neq a \in H^1(P^n; \mathbb{Z}_2)\).

That is, \(H^*(P^n; \mathbb{Z}_2)\) is the unital graded commutative ring generated by a single element \(a\) of degree 1 subject to the relation \(a^{n+1} = 0\).

2. **Stiefel-Whitney Classes**

Axioms. The Stiefel-Whitney classes are cohomology classes \(w_k(\xi) \in H^k(X; \mathbb{Z}_2)\) assigned to each vector bundle \(\xi : E \to X\) such that the following axioms are satisfied:

(S1) \(w_0(\xi) = 1_X\)

(S2) \(w_k(\xi) = 0\) if \(\xi\) is an \(n\)-dimensional vector bundle and \(k > n\)

(S3) naturality: \(w_k(\xi) = f^*(w_k(\eta))\) if there is a bundle map \(\xi \to \eta\) with base map \(f\)

(S4) Whitney product axiom: \(w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi)w_j(\eta)\)

(S5) non-triviality: \(w_1(\gamma_1) \neq 0\) where \(\gamma_1\) is the canonical line bundle over \(P^1\)

If we define the total Stiefel-Whitney class as \(w(\xi) := (w_k(\xi)) \in H^{**}(X; \mathbb{Z}_2)\), then the fourth axiom can be equivalently stated as follows:

(S4') \(w(\xi \oplus \eta) = w(\xi)w(\eta)\)

Observe that every total Stiefel-Whitney class is invertible by axiom (S1) and Lemma 5. If \(w(\xi) = 1\) then we say that the total Stiefel-Whitney class is trivial.
Finally we define the Stiefel-Whitney classes of a manifold $M$ in terms of its tangent bundle:

$$w_i(M) := w_i(\tau_M) \quad \text{and} \quad w(M) := w(\tau_M)$$

**Consequences.** The following properties follow directly from the axioms.

8 **Proposition.** Two equivalent vector bundles have equal Stiefel-Whitney classes.

*Proof.* A bundle equivalence is a bundle map inducing the identity as its base map; hence the claim follows by naturality (S3).

9 **Proposition.** Trivial vector bundles have trivial total Stiefel-Whitney class.

*Proof.* Any trivial vector bundle over $X$ is equivalent to a vector bundle of the form $X \times \mathbb{R}^n$. Thus the diagram

$$
\begin{array}{ccc}
X \times \mathbb{R}^n & \xrightarrow{\text{proj}_2} & \mathbb{R}^n \\
\downarrow{\text{proj}_1} & & \downarrow \\
X & \rightarrow & \{*\}
\end{array}
$$

shows that there is a bundle map to a vector bundle over a point. But we know that all higher cohomology groups of a point are trivial. Hence the claim follows from naturality (S3).

Recall that a manifold is called *parallelizable* if its tangent bundle is trivial. Thus the Stiefel-Whitney classes of a manifold measure obstruction to parallelizability:

10 **Corollary.** Parallelizable manifolds have trivial Stiefel-Whitney class.

11 **Corollary (Stability).** The Stiefel-Whitney classes do not change upon addition of a trivial bundle.

*Proof.* Apply the Whitney product axiom (S4).

12 **Corollary.** Let $\xi$ be an $n$-dimensional Euclidean bundle. If $\xi$ possesses $k$ nowhere linearly dependent sections, then the Stiefel-Whitney classes $w_i(\xi)$ vanish already if $i > n - k$.

*Proof.* The $k$ nowhere linearly dependent sections span a *trivial* $k$-dimensional subbundle $\eta \subseteq \xi$. By means of the continuous inner product we can construct a $(n-k)$-dimensional orthogonal complement $\eta^\perp$. We have $\eta \oplus \eta^\perp = \xi$, hence $w(\xi) = w(\eta^\perp)$ by stability, and the claim follows from axiom (S2).

**Tangent and Normal Bundle.** Let $N \subseteq M$ be a smooth submanifold of a smooth Riemannian manifold. Then the tangent bundle of $N$ is a subbundle of the tangent bundle of $M$ restricted to $N$, i.e. $\tau_N \subseteq \tau_M|N$ and, as before, by means of the smooth inner product we can construct an orthogonal complement $\nu^M_N$ called the **normal bundle** of $N \subseteq M$, and $\tau_N \oplus \nu^M_N \cong \tau_M|N$.

13 **Corollary (Whitney Duality Theorem).** If $N \subseteq M$ is a smooth submanifold of a manifold in Euclidean space (i.e. $M \subseteq \mathbb{R}^n$ open with the identity chart), then

$$w(N) = w(\nu^M_N)^{-1}$$

**Example 14.** The tangent bundle of a sphere has trivial total Stiefel-Whitney class.
Proof. The normal bundle for the standard embedding $S^n \subseteq \mathbb{R}^{n+1}$ is trivial. □

In particular, the tangent bundle of a sphere cannot be distinguished from a trivial bundle over the sphere by means of their Stiefel-Whitney classes.

3. Bundles over Projective Space

Canonical Line Bundle. We will now calculate the Stiefel-Whitney classes of the canonical line bundles directly from the axioms:

15 Proposition. The total Stiefel-Whitney class of the canonical line bundle $\gamma_1^n$ over $P^n$ is given by

$$w(\gamma_1^n) = 1 + a$$

where $a$ denotes the generator of $H^*(P^n; \mathbb{Z}_2)$ (cf. Thm. [7]).

Proof. We have an obvious bundle map

$$
\begin{array}{ccc}
E(\gamma_1^n) & \rightarrow & E(\gamma_1^n) \\
\gamma_1^1 & \rightarrow & \gamma_1^n \\
P^1 & \rightarrow & P^n \\
\end{array}
$$

Therefore

$$0 \lll (SS) \lll w_1(\gamma_1^n) = \text{incl}^*(w_1(\gamma_1^n))$$

and this shows that $w_1(\gamma_1^n) = a$, hence $w(\gamma_1^n) = 1 + a$ since the bundle is one-dimensional. □

Tangent Bundle. By definition the canonical line bundle over $P^n$ is a subbundle of the trivial $(n + 1)$-dimensional bundle. We can thus consider its orthogonal complement $\gamma_K^n$ which is given by

$$E(\gamma_K^n) := \{([x], v) \in P^n \times \mathbb{R}^{n+1} : x \perp v\} \xrightarrow{\text{proj}} P^n$$

16 Proposition. The total Stiefel-Whitney class of the orthogonal complement bundle is given by

$$w(\gamma_K^n) = 1 + a + a^2 + \ldots + a^n$$

Proof. Since the Whitney sum $\gamma_1^n \oplus \gamma_K^n = e^{n+1}$ is trivial (by construction) we have

$$w(\gamma_K^n) = w(\gamma_1^n)^{-1} = (1 + a)^{-1} = 1 + a + a^2 + \ldots + a^n$$

In particular, this shows that all of the first $n$ Stiefel-Whitney classes of an $n$-dimensional vector bundle may be non-zero.

17 Proposition. The tangent bundle $\tau_{P^n}$ of $P^n$ is equivalent to the homomorphism bundle $\text{Hom}(\gamma_1^n, \gamma_1^n)$.
Proof. Recall that the tangent bundle of projective space can be defined as follows:
\[ TP^n := (TS^n := \{(x, v) \in S^n \times \mathbb{R}^{n+1} : x \perp v\})/\{\pm 1\} \mapsto S^n/\{\pm 1\} =: P^n \]
Note that every point of \( S^n \) naturally represents a point in the canonical line bundle \( \gamma^1_n \) and every point of the tangent plane of \( S^n \) naturally represents a point in the orthogonal complement bundle \( \gamma^1_n \). This suggest defining a map
\[ TP^n \to \text{Hom}(\gamma^1_n, \gamma^1_n), \quad [(x, v)] \mapsto (x \mapsto v) \]
and it is straightforward to verify that all equivalence relations fit together in such a way that this map is a well-defined bundle equivalence. □

18 Theorem. The following bundles are equivalent:
\[ \tau_{P^n} \oplus \epsilon^1 \cong \bigoplus_{k=1}^{n+1} \gamma^1_n \]
In particular, the total Stiefel-Whitney class of projective space is given by
\[ w(P^n) = (1 + a)^{n+1} \]
Proof. The endomorphism bundle \( \text{Hom}(\gamma^1_n, \gamma^1_n) \) is trivial (consider the non-vanishing identity section). Thus
\[ \tau_{P^n} \oplus \epsilon^1 \cong \text{Hom}(\gamma^1_n, \gamma^1_n) \oplus \text{Hom}(\gamma^1_n, \gamma^1_n) \cong \text{Hom}(\gamma^1_n, \bigoplus_{k=1}^{n+1} \gamma^1_n) \]
\[ \cong \text{Hom}(\gamma^1_n, \epsilon^{n+1}) \cong \bigoplus_{k=1}^{n+1} \gamma^1_n \]
where the last equivalence is induced by the continuous inner product of the Euclidean bundle \( \gamma^1_n \). □

19 Corollary (Stiefel). The total Stiefel-Whitney class of the projective space \( P^n \) is trivial if and only if \( n+1 \) is a power of 2.
Proof. Write \( n+1 = 2^k m \) with odd \( m \). By the Frobenius homomorphism we have
\[ w(P^n) = (1 + a)^{n+1} = (1 + a^{2^k})^m = 1 + a^{2^k} + \binom{m}{2} a^{2^{k+1}} + \ldots \]
It follows that \( w(P^n) \) is trivial if and only if \( 2^k = n+1 \). □

Applications. A (not necessarily associative) algebra is called a division algebra if every equation of the form \( ax = b \) and \( xa = b \) with nonzero \( a \) and arbitrary \( b \) has a unique solution.

20 Lemma. A finite-dimensional algebra is a division algebra if and only if it has no zero divisors.
Proof. Left and right multiplication are linear endomorphisms of a finite-dimensional vector space, hence injective if and only if surjective. □

21 Theorem (Stiefel). Suppose there exists a real division algebra of dimension \( n \). Then the projective space \( P^{n-1} \) is parallelizable and \( n \) is a power of 2.
Proof. Up to isomorphism, any real division algebra of dimension \( n \) is of the form \( (\mathbb{R}^n, +) \) with a bilinear product \( p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) without zero divisors. Let us
denote the standard basis of \( \mathbb{R}^n \) by \((e_i)\). Since \( p \) has no zero divisors, it induces automorphisms \( p(\cdot, e_i) \) and
\[
v_i := p(\cdot, e_i)p(\cdot, e_1)^{-1}
\]
Note that \( v_1 = \text{id} \) and \((v_i(x))\) are linearly independent for \( x \neq 0 \):
\[
\sum \lambda_i v_i(x) = 0 \Rightarrow p(x, \sum \lambda_i e_i) = 0 \Rightarrow \lambda_i = 0 \text{ or } x = 0
\]
We can thus define sections of the bundle \( \text{Hom}(\gamma_{n-1}^1, \gamma_{n-1}^1) \cong \tau_{P^{n-1}} \) as follows:
\[
s_i([x])(y) := \text{orthogonal projection of } v_i(y) \text{ along } \langle x \rangle
\]
Since \( s_1 = 0 \) it follows that \( s_2, \ldots, s_n \) are \( n-1 \) nowhere linearly dependent sections. We have thus proved that \( P^{n-1} \) is parallelizable, and now Cor. \([19]\) shows that \( n \) must be a power of 2.

In fact, one can show that the projective space \( P^{n-1} \) is parallelizable only for \( n = 1, 2, 4 \) or 8. It follows that finite-dimensional real division algebras exist precisely in these dimensions!

References


