Ziel

MTC definieren: $SL_2(\mathbb{Z}) \overset{\text{proj.}}{\curvearrowright} Hom(\mathbb{1}, H)$ ~~proj~~.

einfach — kanonisches Objekt

Wiederholung

Verzopfte Tensorkategorie (BTC): (Abelsche) monoidale Kategorie mit nat. Kommutativitätsisos $(\sigma_{A,B})$

Rigide Kategorie: (Abelsche) monoidale Kategorie mit rechten und linken Dualen

rechter Dual von V: V^* mit $e_V: V^* \otimes V \longrightarrow \mathbb{1}$, $i_V: \mathbb{1} \longrightarrow V \otimes V^*$
so dass $V \longrightarrow V \otimes V^* \otimes V \longrightarrow V = id_V$ und analog für V^*.

Dort:

1) $Hom(U \otimes V, W) \cong Hom(U, W \otimes V^*)$
$Hom(U, V \otimes W) \cong Hom(V^* \otimes U, W)$

2) Insbesondere:
$Hom(U, V) \cong Hom(\mathbb{1}, V \otimes U^*) \cong Hom(V^*, U^*)$

$\rightsquigarrow$ Kategoriendualität $(-)^*$ (kompatibel mit MK (BTC)-Struktur)

3) $\otimes$ ist exakt und die Grothendieckgruppe

$K(C) := \{\text{Isoklassen}\} / \{\langle A \rangle + \langle C \rangle = \langle B \rangle : 0 \to A \to B \to C \to 0 \text{ exakt}\}$

exakt gdw mit $Hom(Y, -)$ exakt

linksexakter Funktor (V)

wird zum Ring mit

$\langle A \rangle \cdot \langle B \rangle := \langle A \otimes B \rangle$.

<u>Bandkategorie</u> (ribbon category, RC): Rigide BTC mit Twists ($\theta_V : V \xrightarrow{\cong} V$) mit

$$\theta_1 = id_1, \quad \theta_{V\otimes W} = \sigma_{W,V}\,\sigma_{V,W}\,(\theta_V \otimes \theta_W), \quad \theta_{V^*} = (\theta_V)^*$$

($\hat{=}$ $\delta_V : V \to V^{**}$ mit ...)

$\leadsto$ <u>Spur</u> eines Endos $f : V \to V$:

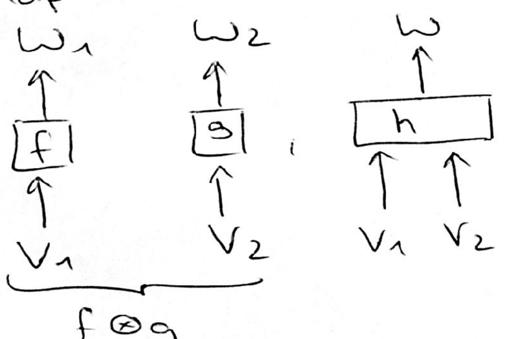
$$tr(f)\cdot id_1 := 1 \xrightarrow{i_V} V \otimes V^* \xrightarrow{f \otimes id} V \otimes V^* \xrightarrow{\delta_V \otimes id} V^{**} \otimes V^* \xrightarrow{ev_{V^*}} 1$$

und <u>Dimension</u>

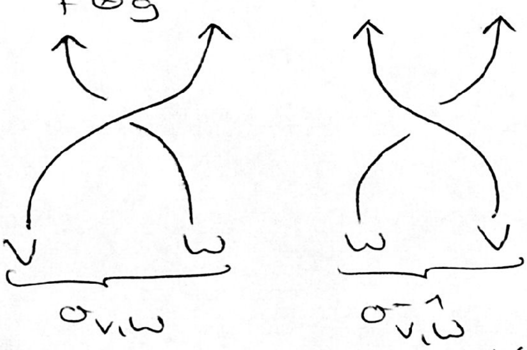
$$\dim(V) := tr(id_V).$$

<u>Grafischer Kalkül</u>:

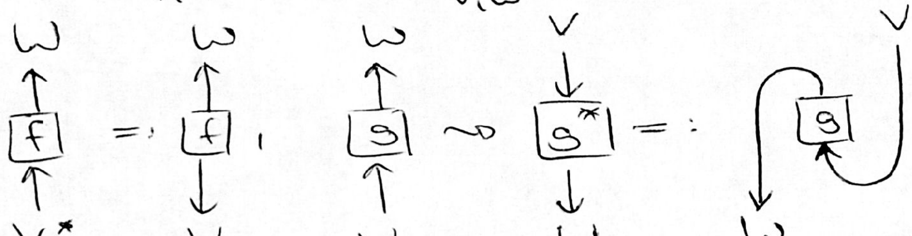
für Morphismen



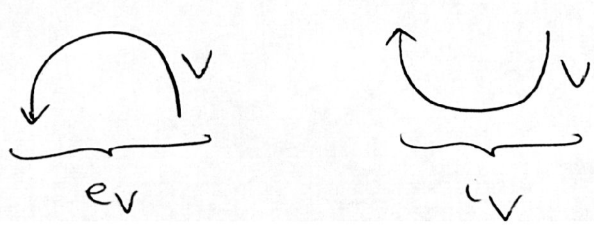
, 1 weglassen (MK)



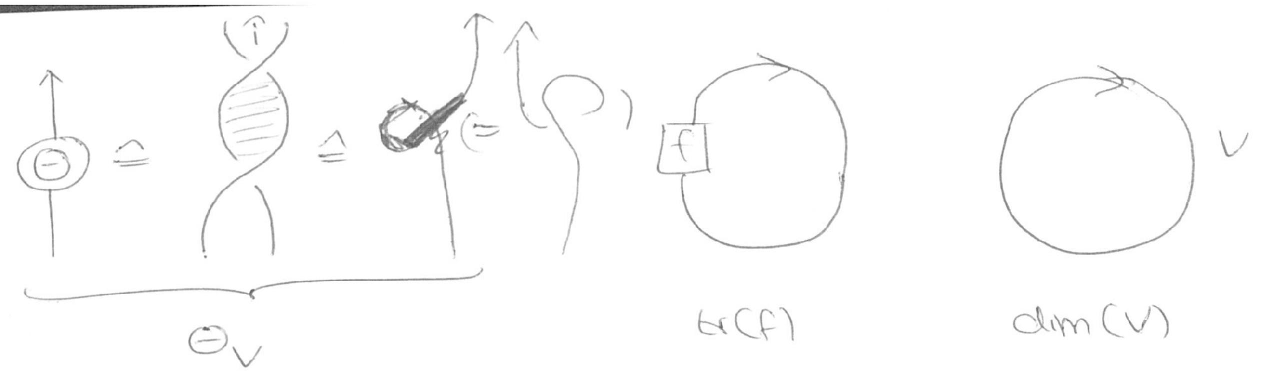
(BTC)



(Rigide)



B



(RC)

<u>Satz 1</u> (Reshetikhin-Turaev): Der zugehörige Morphismus hängt nur von der Isotopieklasse des Bandes ab.

Halbeinfache Kategorien

Ein Objekt V $(\neq 0)$ heißt einfach, falls jeder Mono $U \hookrightarrow V$ 0 oder ein Iso ist.

(~~Äquivalent~~: Jeder ~~Epi~~ Epi $V \twoheadrightarrow W$ ist 0 oder ein ~~Mono~~ Iso.)

$\rightsquigarrow$ ~~Fast~~

Bew: Unterobjekte $\cong$ Quotienten
$\mapsto$ coker
ker $\leftarrow$

Eine abelsche Kategorie heißt halbeinfach, falls sich jedes Objekt V in der Form

$$V \cong \bigoplus_{i \in I}^{\text{endl}} N_i V_i$$

schreiben lässt; dabei: (V_i) Repr. system der Isoklassen einfacher Objekte. Wir fordern zusätzlich

$$\operatorname{End}(V_i) \cong k$$

$\rightsquigarrow$ $\operatorname{Hom}(V_i, V_j) = \delta_{ij} k$, (N_i) eind. bestimmt $(N_i = \dim \operatorname{Hom}(V_i, V))$

Sei nun $\mathcal{C}$ halbeinfache Bandkategorie.

1) Konvention: $V_0 = \mathbb{1}$.

2) V_i einfach $\Rightarrow$ $V_i^* =: V_{i^*}$ einfach

$\rightsquigarrow$ Involution $(-)^*: I \to I$ mit $0^* = 0$.

3)
$$V_i \otimes V_j =: \bigoplus_{k \in I} N_{ij}^k V_k \qquad (\#)$$

mit

- $N_{ij}^k = \dim \operatorname{Hom}(V_k, V_i \otimes V_j) = \dim \operatorname{Hom}(\mathbb{1}, V_i \otimes V_j \otimes V_{k^*})$
- $N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^* j^*}^{k^*}$

  (#) dualisieren

- $N_{i0}^k = \delta_{ik}$

4) Für

$\Theta_{V_i} =: \Theta_i \cdot \mathrm{id}_{V_i}$

$d_i := \dim(V_i) = \mathrm{tr}(\mathrm{id}_{V_i})$

gelten:

- $\Theta_0 = 1, \quad \Theta_{i^*} = \Theta_i$
- $d_0 = 1, \quad d_{i^*} = d_i, \quad d_i d_j = \sum N_{ij}^k d_k$

<u>Lemma 2</u>: $d_i \neq 0 \ (\forall i)$

<u>Beweis</u>: Es gilt:

$N_{i,i^*}^{0} = N_{i,0}^{i} = 1$

$\rightsquigarrow$ schreibe

$V_i \otimes V_i^* \cong: 1 \oplus W$

sowohl Koprodukt als auch Produkt in einer abelschen Kategorie $\Big\}$ „Biprodukt“

mit $\mathrm{Hom}(1, W) = 0$.

$\rightsquigarrow \mathrm{Hom}(W, 1) = 0$.

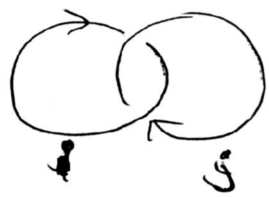
Betrachte

$$\begin{tikzcd}
1 \arrow[r, "c_{V_i}", "\neq 0"'] \arrow[drr, "f \oplus 0"] \arrow[ddrr, "f"'] & V_i \otimes V_i^* \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, leftrightarrow, "\cong"] & & V_i^{**} \otimes V_i^* \arrow[r, "e_{V_i^*}", "\neq 0"'] \arrow[dl, leftrightarrow, "\cong"] & 1 \\
& & 1 \oplus W \arrow[urr] \arrow[d, Leftrightarrow] & & \\
& & 1 \arrow[uurr, "g"'] & &
\end{tikzcd}$$

$\Rightarrow f, g \neq 0$ in $k \cong \mathrm{End}(1, 1)$

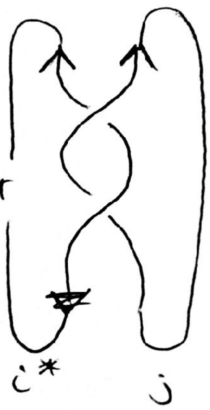
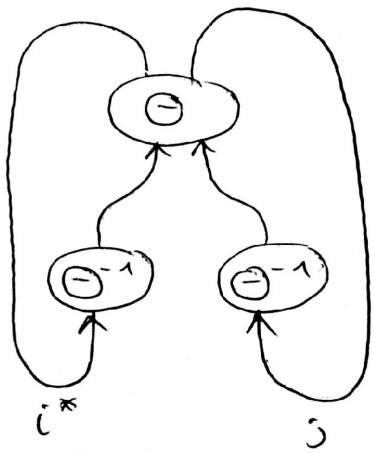
$\Rightarrow d_i = f \cdot g \neq 0$. $\square$

<u>Modulare Tensorkategorie</u> (MTC): Halbeinfache Bandkategorie mit $\#I<\infty$ und in der die Matrix $\tilde{S}:=(\tilde{S}_{ij})$ mit

$\tilde{S}_{ij} =$ 

invertierbar ist.

Es gilt:

$\tilde{S}_{ij} =$  $\overset{!}{=}$  $= \theta_i^{-1}\theta_j^{-1}\,\mathrm{tr}\Big($  $\Big)$

$$= \theta_i^{-1}\theta_j^{-1}\sum_k N_{i^*j}^{k}\,\mathrm{tr}(\theta_{V_k}) = \theta_i^{-1}\theta_j^{-1}\sum_k N_{i^*j}^{k}\,\theta_k d_k$$

Insbesondere:

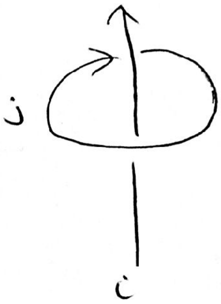
- $\tilde{S}_{ij} = \tilde{S}_{ji} = \tilde{S}_{i^*j^*}$
- $\tilde{S}_{i0} = \theta_i^{-1}\sum \delta_{i^*k}\,\theta_k d_k = d_i$.

<u>Bem</u>: Ist $\mathcal{C}$ symmetrisch, d.h.  $=$ , dann

$\tilde{S}_{ij} = d_i d_j$

$\rightsquigarrow \tilde{S}$ singulär für $\#I>1$ $\lightning$

Lemma 3:



$= \dfrac{\tilde{S}_{ij}}{d_i}$

Bew:

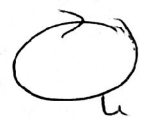
$\mathrm{tr}: \mathrm{End}(V_i) \xrightarrow{\cong} k, \quad c \cdot \mathrm{id}_{V_i} \longmapsto c \cdot d_i$

und

$\mathrm{tr}(\text{L.H.S.}) = \tilde{S}_{ji} = \tilde{S}_{ij} = \mathrm{tr}(\text{R.H.S.})$ $\square$

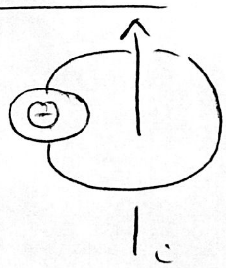
Konvention:

$:= \sum d_k$



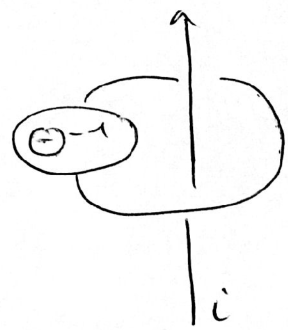
$= \sum d_k$

Lemma 4:



$= p^+$

und



$= p^-$

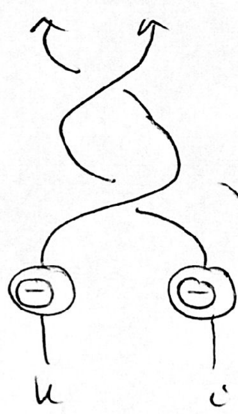
mit $p^{\pm} = \sum \Theta_k^{\pm 1} d_k$.

Bew (nur „+"): Wir zeigen:

$\mathrm{tr}(\text{L.H.S.} \overset{\leftrightarrow}{\cdot} \Theta_i) = p^+ d_i$

Tatsächlich:

$\mathrm{tr}(\Theta_i \cdot \text{L.H.S.}) = \sum d_k \, \mathrm{tr}($

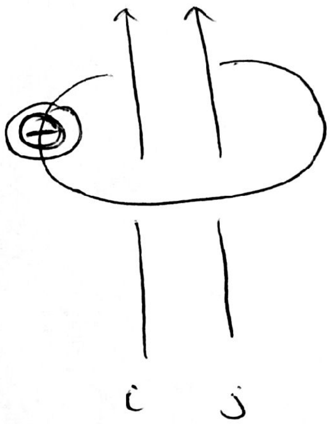


$) = \sum d_k \, \mathrm{tr}($

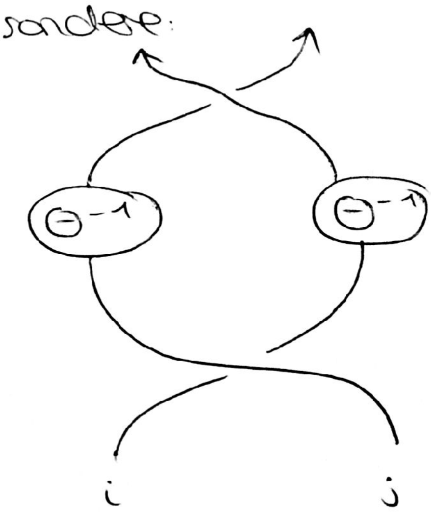
$)$

$$= \sum_k d_k \sum_j N_{ik}^j \, \Theta_j d_j = \sum_j d_i d_j^2 \Theta_j = p^+ d_i$$ $\square$

Korollar 5: Das Lemma gilt für beliebige V anstatt nur V_i. Insbesondere:



$= p^+$



Bew: $V = V_i \otimes V_j$ und

RHS $= p^+$

.

□

$SL_2(\mathbb{Z}) \overset{\text{proj}}{\curvearrowright} k^I$

$$\underline{SL_2(\mathbb{Z})} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : ad - bc = 1 \right\}$$

$$= \langle s, t \mid (st)^3 = s^2,\ s^4 = 1 \rangle$$

$$\uparrow$$

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

<u>Theorem 6</u>: Für $\tilde{S} = (\tilde{S}_{ij})$, $t := (t_{ij}) := (\delta_{ij}\Theta_i)$, $c = (c_{ij}) := (\delta_{ij^*})$ gelten:

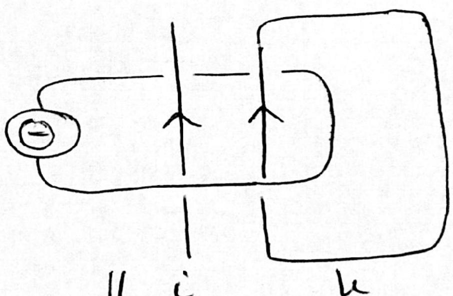
1) $(\tilde{S}t)^3 = p^+ \tilde{S}^2$

2) $(\tilde{S}t^{-1})^3 = p^- \tilde{S}^2 c$

3) $ct = tc$, $c\tilde{S} = \tilde{S}c$, $c^2 = 1$

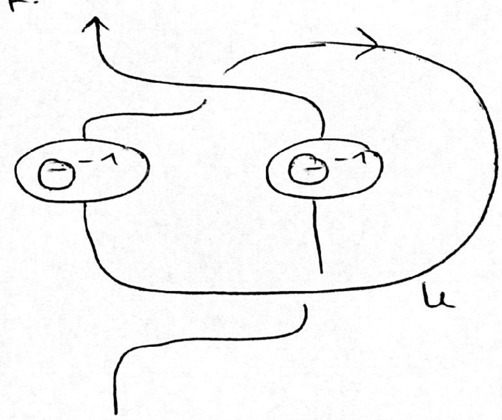
4) $\tilde{S}^2 = p^+ p^- c$, falls $\tilde{S}$ invertierbar

<u>Bew</u>: 3) ✓ 1) Korollar liefert:



i $\quad$ k

$\overset{⑤}{=}$ p^+

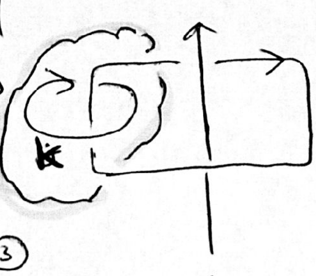


k

i

$\|$

$\sum d_j \Theta_j$

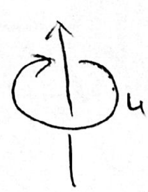


k $\quad$ j

i

$\|$ ③

$p^+ \Theta_i^{-1} \Theta_k^{-1}$



k

i

③ $\|$ ③

$$\sum d_j \Theta_j \frac{\tilde{S}_{jk} \tilde{S}_{ij}}{d_i d_j}$$

$\|$ ③

$$p^+ \Theta_i^{-1} \Theta_k^{-1} \frac{\tilde{S}_{ik}}{}$$

$$\Rightarrow \sum_j \tilde{S}_{ij}\,\Theta_j\,\tilde{S}_{jk} = p^+\,\Theta_i^{-1}\,\tilde{S}_{ik}\,\Theta_k^{-1}$$

d.h. $\tilde{s}t\tilde{s} = p^+ t^{-1}\tilde{s}t^{-1}$

$$\Rightarrow (\tilde{s}t)^3 = \tilde{s}t(\tilde{s}t\tilde{s})t = p^+\tilde{s}^2.$$

2) analog mit $\tilde{s}t^{-1}\tilde{s} = p^- t\tilde{s}tc.$

4) $\underline{\tilde{s}t\tilde{s}}^2 = p^+ t^{-1}\underline{\tilde{s}t^{-1}\tilde{s}} = p^+p^-\tilde{s}tc$

$$\Rightarrow \tilde{s}^2 = p^+p^-c. \qquad \square$$

Insbesondere: In einer MTC gelten $p^+, p^- \neq 0$. Setze

$$D := \sqrt{p^+p^-},\quad \zeta := \left(\frac{p^+}{p^-}\right)^{\frac{1}{6}} \quad (\in \overline{k})$$

$$s := \frac{\tilde{s}}{D}$$

Dann gelten:

- $(st)^3 = \frac{p^+}{D}s^2 = \zeta^3 s^2$
- $s^4 = 1$

$$\rightsquigarrow SL_2(\mathbb{Z}) \overset{\text{proj}}{\circlearrowright} \overline{k}^I$$

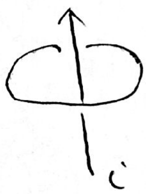
Verlinde-Algebra und -Formel

Verlinde-Algebra einer MTC $\mathcal{C}$.

$K := K(\mathcal{C}) \otimes_{\mathbb{Z}} k$.

$\rightsquigarrow$ endl.dim, Basis $x_i = \langle V_i \rangle$, Einselement $1 = x_0 = \langle 1 \rangle$.

Korollar 7: Es gelten:

1) 

$= p^+p^- \delta_{i,0}$ 

2) $p^+p^- = \sum d_i^2 =$  .

Bew:

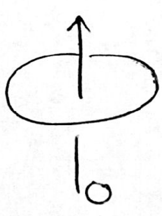
1)

~~tr(LHS)~~

$$tr(LHS) = \sum_j d_j \tilde{S}_{i,j} = \sum_j \tilde{S}_{0,j} \tilde{S}_{j,i} = (\tilde{S}^2)_{0,i}$$

$$= p^+p^- C_{0,i} = \delta_{i,0} p^+p^- = tr(RHS).$$

2)

$p^+p^- \overset{1)}{=} tr$ 

$= \sum d_i \tilde{S}_{i,0} = \sum d_i^2$

oder

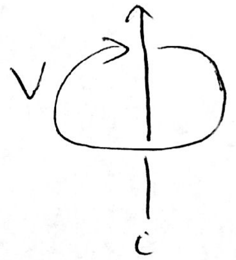
$$p^+p^- = p^+p^- C_{0,0} = (\tilde{S}^2)_{0,0} = \sum_i d_i^2. \qquad \square$$

$(\rightsquigarrow \tilde{S}_{0,0}^{-1} = D = \sqrt{\sum d_i^2})$

Theorem 8: Der Algebrenhom

$$\mu : K \longrightarrow \mathrm{Map}(I, k)$$

def. durch



$$=: \mu(\langle V \rangle)(i)$$

ist ein Isomorphismus.

Bew: Alghom ✓ Iso: Def. Basis

$$\varepsilon_i : I \longrightarrow k, \quad \varepsilon_i(j) := \frac{\delta_{i,j}}{S_{0,i}}$$

$$\Rightarrow \mu(x_j)(i) = \frac{\tilde{S}_{i,j}}{\tilde{S}_{i,0}} = \frac{S_{i,j}}{S_{i,0}}$$

d.h. $\mu(x_j) = \sum \underbrace{S_{i,j}}_{\text{invertierbar.}} \varepsilon_i$ □

$\rightsquigarrow$ bzgl. $\hat{\varepsilon}_i := \mu^{-1}(\varepsilon_i)$ ist die Multiplikation in K diagonal!

Genauer: Betrachte

$$x_i \hat{\varepsilon}_j = \frac{S_{i,j}}{S_{0,j}} \hat{\varepsilon}_j \quad \rightsquigarrow (D_i)_{a,b} := \delta_{a,b} \frac{S_{i,a}}{S_{0,a}}$$

$$x_i x_j = \sum N_{i,j}^{k} x_k \quad \rightsquigarrow (N_i)_{a,b} := N_{i,b}^{a}$$

Dann gilt:

$$\boxed{S N_i S^{-1} = D_i}$$

Theorem 9 (Verlinde-Formel):

$$N_{i,j}^{k} = \sum_r \frac{S_{i,r}\, S_{j,r}\, S_{k^*,r}}{S_{0,r}} \qquad (\in \mathbb{N}_{\geq 0})$$

Beweis:

$$S N_i = D_i S$$

$$\Rightarrow \sum_a S_{r,a} (N_i)_{a,j} = \sum_a (D_i)_{r,a} S_{a,j}$$

$$\sum_a S_{r,a} N_{i,j}^{a} \qquad\qquad \sum_a \delta_{r,a} \frac{S_{i,a}}{S_{0,a}} S_{a,j} = \frac{S_{i,r}\, S_{j,r}}{S_{0,r}}$$

$$\Rightarrow \text{RHS} = \sum_{a,r} S_{r,a} N_{i,j}^{a} S_{k^*,r} = \sum_a N_{i,j}^{a} \underbrace{(S^2)_{a,k^*}}_{=\delta_{a,k}} = N_{i,j}^{k}.$$

$\square$

via $\bigoplus e_i \otimes (e_i^*)^{-1}$

$\underline{SL_2(\mathbb{Z}) \overset{\text{proj}}{\curvearrowright} \operatorname{Hom}(\mathcal{U}, H)}$

einfach

kanonisches Objekt

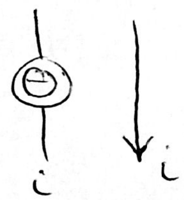
$$H := \bigoplus V_i \otimes V_i^* \qquad (\cong H^* \cong \bigoplus V_i^* \otimes V_i)$$

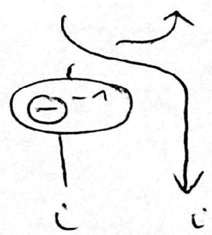
$\rightsquigarrow \dim H = \sum d_i^2 = D^2$

<u>Theorem 10</u>: Für die Endos $S := \bigoplus S_{ij}$, $T := \bigoplus T_{ij}$,

$C := \bigoplus C_{ij}$ von H def. durch

- $S_{ij} = \frac{d_i}{D}$ 

$T_{ij} := \delta_{ij}$ 

$C_{ij} := \delta_{ij^*}$ 

- gilt:

- $S^2 = C$
- $C^2 = \Theta_H^{-1}$ <u>nicht</u> const. id_H ↯
- $(ST)^3 = \sqrt{\frac{p^+}{p^-}}\, S^2$

$\rightsquigarrow$ betrachte $\operatorname{Hom}(\mathcal{U}, H)$ für $\mathcal{U}$ einfach!

Korollar 11: Für U einfach gilt:

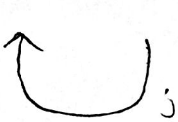
$$\Theta_H \circ - \big|_{\mathrm{Hom}(U,H)} =: \underset{\substack{\cap \\ k}}{\Theta_U} \cdot \mathrm{id}_{\mathrm{Hom}(U,H)}$$

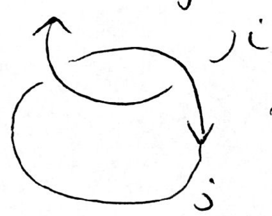
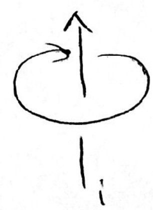
$\rightsquigarrow SL_2(\mathbb{Z}) \overset{\text{proj}}{\curvearrowright} \mathrm{Hom}(U,H)$ durch Postkomposition mit S, T.

Beispiel ($U = \mathbb{1}$):

$$\mathrm{Hom}(\mathbb{1}_{\mathcal{H}}, H) \cong \bigoplus \mathrm{End}(V_i) \cong k^I$$

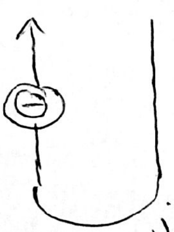
kanonische Basis via ev_{V_j} 

$x_j =$  $\cong$ id_{V_j} $\uparrow_j \cong (\delta_{ij})$

$\Rightarrow \underline{S_{ij} \circ x_j} = \frac{d_i}{D}$  $\cong \frac{d_i}{D}$ 

$= \frac{\tilde{s}_{ij}}{D} \uparrow_i \cong \underline{S_{ij} \cdot x_i}$

und

$\underline{T_{ij} \circ x_j} = \delta_{ij}$  $\cong \delta_{ij}$  $= \underbrace{\delta_{ij}\,\theta_j}_{= t_{ij}} \uparrow_j$

$\cong \underline{t_{ij} \cdot x_j}$.