THE GEOMETRIC GROUP OF THE BAUM-CONNES CONJECTURE FOR DISCRETE GROUPS

MICHAEL WALTER

Notation and Conventions. Let Γ be a countable discrete group. We will assume that all spaces are metrizable (in particular, Hausdorff and paracompact).

1. Proper actions

Proper Γ-spaces. Let X be a Γ-space. X is called *proper* if the orbit space $\Gamma\backslash X$ is Hausdorff and every point $x \in X$ has a Γ-invariant open neighborhood U with a Γ-map $U \to \Gamma/H$ for some *finite* subgroup $H \subseteq \Gamma$.

Example 1. Add Łukasz' examples...

The following lemma is central in proving important properties about proper Γ-spaces:

Technical Lemma 2 ([BCH94, Appendix 1, Lemma])**.** *Let X be a proper Γ-space. Then we can find a countable partition of unity (σ_k) of Γ-invariant functions and Γ-maps*

$$\mathrm{supp}(\sigma_k) \xrightarrow{\phi_k} \coprod_{H \subseteq \Gamma \text{ finite}} \Gamma/H$$

We can use it to show that every proper Γ-space is also proper in the usual sense:

3 Lemma. Let X is a proper Γ-space and $K, L \subseteq X$ compact. Then the set

$$\{\gamma \in \Gamma : \gamma K \cap L \neq \varnothing\}$$

is finite.

Proof. From the Technical Lemma we get a Γ-map

$$\phi : X \to \coprod_{H \subseteq \Gamma \text{ finite}} \Gamma/H, \; x \mapsto \sum \sigma_k(x)\phi_k(x)$$

Noting that the target space is discrete we see that the images of K and L under this map are finite. In particular, they are contained in already finitely many summands Γ/H_i. Hence the above set contains at most

$$\#\phi(K) \cdot \#\phi(L) \cdot \max_i H_i$$

group elements. ∎

The preceding lemma in turn can be used to prove many other results about proper spaces.

Date: Last compiled January 28, 2011; last modified August 28, 2009 or later.

4 Lemma. Let X be a proper Γ-space and $f \in C_c(X)$. Then

$$||\sum_{\gamma\in\Gamma} \gamma f||_\infty < \infty$$

where the sum is understood as a pointwise limit.

Proof. If $x \in \operatorname{supp}(\gamma_0 f)$ for some $\gamma_0 \in \Gamma$ then

$$|\sum_{\gamma\in\Gamma}(\gamma f)(x)| = |\sum_{\gamma\in\Gamma} f(\gamma^{-1}x)| \leqslant ||f||_\infty \cdot \#\{\gamma \in \Gamma : \gamma^{-1}x \in \operatorname{supp}(f)\}$$

But

$$\begin{aligned} &\#\{\gamma \in \Gamma : \gamma^{-1}x \in \operatorname{supp}(f)\} \\ \leqslant &\#\{\gamma \in \Gamma : \gamma^{-1}\operatorname{supp}(\gamma_0 f) \cap \operatorname{supp}(f) \neq \varnothing\} \\ \leqslant &\#\{\gamma \in \Gamma : \gamma^{-1}\operatorname{supp}(f) \cap \operatorname{supp}(f) \neq \varnothing\} < \infty \end{aligned}$$

independent of x. □

A proper Γ-space $\underline{E}\Gamma$ is called *universal* if for every other proper Γ-space X there exists a unique Γ-map $X \to \underline{E}\Gamma$ up to Γ-homotopy. Clearly, any two such spaces are unique up to Γ-homotopy equivalence.

5 Proposition. The space

$$\underline{E}\Gamma := \{f : \Gamma \to [0,1] : f \text{ finitely supported}, \sum_{\gamma\in\Gamma} f(\gamma) = 1\}$$

equipped with the supremum metric and the obvious left Γ-action is a universal proper Γ-space.

Proof. *Properness:* It follows from the finite support condition that the metric "descends" to the quotient space. Hence it is metrizable and, in particular, Hausdorff. Now let $f \in \underline{E}\Gamma$. Since it has finite support we observe that

$$R := \inf\{||f - \gamma f||_\infty : \gamma \in \Gamma \backslash \operatorname{Stab}(f)\} > 0$$

If we choose $\epsilon \in (0, \frac{R}{2})$ then for every element in the open Γ-invariant neighborhood

$$U := \{g \in \underline{E}\Gamma : \exists \gamma_g \in \Gamma : ||g - \gamma_g f|| < \epsilon\}$$

the associated γ_g is uniquely define modulo $\operatorname{Stab}(f)$. Thus we get a Γ-map

$$U \to \Gamma/\operatorname{Stab}(f),\ g \mapsto \gamma_g \operatorname{Stab}(f)$$

Universality: Let X be any proper Γ-space. Evidently, any two Γ-maps $X \to \underline{E}\Gamma$ are Γ-homotopic (using convexity of the latter space). Thus it suffices to construct a single such map: Let us invoke the Technical Lemma again and define the Γ-map

$$\phi : \coprod_{H \subseteq \Gamma \text{ finite}} \Gamma/H \to \underline{E}\Gamma,\ \gamma H \mapsto \frac{1}{\#H} 1_H$$

sending a H-coset to the uniform probability distribution on H. Then

$$X \to \underline{E}\Gamma,\ x \mapsto \sum_k \sigma_k(x)(\phi \circ \phi_k(x))$$

is a Γ-map of the desired form. □

Γ-compactness. A proper Γ-space is called Γ-*compact* if $\Gamma\backslash X$ is compact.

6 Lemma. Every Γ-compact proper Γ-space is locally compact.

Proof. Missing. . . □

2. Equivariant K-homology

The *equivariant K-homology* groups of a locally compact Γ-space X are defined as follows in terms of Kasparov's equivariant KK-theory (cf. [Kas88]):

$$K_*^\Gamma(X) := KK_*^\Gamma(C_0(X), \mathbb{C})$$

We cannot apply this construction to an arbitrary proper Γ-space Y since it will in general *not* be locally compact. However, all its Γ-compact subspaces are locally compact by Lemma 6. Hence we will define its *equivariant K-homology* as the direct limit

$$RK_*^\Gamma(Y) := \varinjlim_{X \subseteq Y \ \Gamma\text{-compact}} K_*^\Gamma(X)$$

The arrows of the directed system are induced by inclusions Γ-compact subspaces $X \subseteq X'$. In order for this to make sense we have to show that any such X is closed in X'. But this follows readily from commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\subseteq} & X' \\ \downarrow & & \downarrow \pi \\ \Gamma/X & \xrightarrow{\subseteq} & \Gamma/X' \end{array}$$

$(X = \pi^{-1}(\Gamma/X))$.

In particular, we can apply this construction to the universal proper Γ-space $\underline{E}\Gamma$. The resulting groups $RK_*^\Gamma(\underline{E}\Gamma)$ are the *geometric groups* of the Baum-Connes conjecture for the discrete group Γ.

Cycles. Recall that cycles for $K_*^\Gamma(X)$ are given by triples (H, π, F) containing the following data

1. H is a Hilbert space with a unitary Γ-representation
2. $\pi : C_0(X) \circlearrowleft H$ *covariant* representation, i.e. $\pi(\gamma f) = \gamma\pi(f)\gamma^{-1}$
3. F is a self-adjoint bounded operator on H

such that

$$[\pi(f), F], \quad \pi(f)(F^2 - 1), \quad \pi(f)[\gamma, F]$$

are compact for all $f \in C_0(X)$, $\gamma \in \Gamma$. In the even case, we also require H to be equipped with a $\mathbb{Z}_2$-grading such that Γ and $C_0(X)$ act as even operators and F is odd.

Simplifications. Since a cycle (H, π, F) is homotopic to its compression to the orthogonal complement of the common kernel of the $C_0(X)$-action we can (and will) always assume that π is a *non-degenerate* representation. If X is a Γ-compact proper space then we can assume even more:

7 Proposition. Let X be a Γ-compact proper space. Then any cycle (H, π, F) for $K_*^\Gamma(X)$ is a compact perturbation of a cycle (H, π, G) where F' is Γ-equivariant and *properly supported*, i.e.

$$\forall g \in C_c(X) : \exists f \in C_c(X) : G\pi(g) = \pi(f)G\pi(g)$$

The main ingredient to proving this proposition is the following construction:

8 Lemma. In the situation of Proposition 7 let $k \in C_c(X)$ a *real-valued* function. Then

$$A_k(F) := \sum_{\gamma \in \Gamma} \gamma\pi(k)F\pi(k)\gamma^{-1}$$

converges strongly to a bounded operator on H (with $||A_k(F)|| \leqslant C_k||F||$). Furthermore, $A_k(F)$ is Γ-equivariant and properly supported.

Proof. For showing strong convergence we may assume that $F \geqslant 0$. Then, since $\Gamma \circlearrowleft H$ by unitaries and $\pi(\gamma k)$ is self-adjoint, every summand

$$\gamma\pi(k)F\pi(k)\gamma^{-1} = \pi(\gamma k)\gamma F\gamma^{-1}\pi(\gamma k)$$

is also positive. Hence by Lemma 9 it suffices to establish a uniform norm bound for the partial sums: By conjugating $F \leqslant ||F||$ we find that

$$\pi(\gamma k)\gamma F\gamma^{-1}\pi(\gamma k) \leqslant \pi(\gamma k^2)||F||,$$

hence for every finite subset $\Gamma' \subseteq \Gamma$ we get the estimate

$$||\sum_{\gamma\in\Gamma'} \pi(\gamma k^2)|| \leqslant ||\sum_{\gamma\in\Gamma'} \gamma k^2||_\infty \leqslant ||\sum_{\gamma\in\Gamma} \gamma k^2||_\infty =: C_k \overset{4}{\leqslant} \infty$$

We conclude that $A_k(F)$ exists as a strong limit and that its norm is bounded by $C_k||F||$. Since $A_k(F)$ is positive this holds for any self-adjoint operator F.

It remains to show that $A_k(F)$ it Γ-equivariant and properly supported. The former is evident from the definition. For the latter, let $g \in C_c(X)$. We have

$$A_k(F)\pi(g) = \sum_{\gamma\in\Gamma} \pi(\gamma k)\gamma F\pi(k \cdot (\gamma^{-1}g))\gamma^{-1}$$

As X is a proper Γ-space, the set

$$S := \{\gamma \in \Gamma : k \cdot (\gamma^{-1}g) \neq 0\} = \{\gamma \in \Gamma : \operatorname{supp}(k) \cap \gamma^{-1}\operatorname{supp}(g) \neq \varnothing\}$$

is finite. Hence $V := \bigcup_{\gamma\in S} \operatorname{supp}(\gamma k)$ is compact and we can find a function $f \in C_c(X)$ equal to 1 on V. Any such function satisfies $\pi(f)A_k(F)\pi(g) = A_k(F)\pi(g)$. □

Proof of Proposition 7. By [Bou63, 7.2.4, Proposition 8] there exists a function $0 \leqslant k \in C_c(X)$ with

$$\sum_{\gamma\in\Gamma} k^2(\gamma x) = 1 \quad (\forall x \in X)$$

Note that its partial sums define an approximate unit in $C_0(X)$. Thus it follows from the fact that we assumed π to be non-degenerate that

$$\sum_{\gamma \in \Gamma} \gamma \pi(k^2) \gamma^{-1} F = F$$

(strong convergence). Let us show that $G := A_k(F)$ is a compact perturbation of the operator F. Indeed, for every f in the dense subset $C_c(X) \subseteq C_0(X)$ we get

$$\begin{aligned}
&\pi(f)(F - G) \\
=& \sum_{\gamma \in \Gamma} \pi(f) \gamma \pi(k^2) \gamma^{-1} F - \gamma \pi(k) F \pi(k) \gamma^{-1} \\
=& \sum_{\gamma \in \Gamma} \pi(f) \gamma \pi(k) \left(\pi(k) \gamma^{-1} F - F \pi(k) \gamma^{-1} \right) \\
=& \sum_{\gamma \in \Gamma} \pi(f) \gamma \pi(k) \left(\pi(k) [\gamma^{-1}, F] + [\pi(k), F] \gamma^{-1} \right) \\
=& \sum_{\gamma \in \Gamma} \gamma \pi((\gamma^{-1} f) \cdot k) \left(\pi(k) [\gamma^{-1}, F] + [\pi(k), F] \gamma^{-1} \right)
\end{aligned}$$

Observe that every summand is compact. Thus it follows from

$$\#\{\gamma \in \Gamma : \operatorname{supp}(\gamma^{-1} f) \cap (\operatorname{supp} k) \neq \emptyset\} < \infty$$

being finite that $\pi(f)(F - G)$ is in fact a finite sum of compact operators, hence compact. □

Appendix A. Functional analysis

9 Lemma. Any increasing sequence $(A_n) \subseteq B(H)$ of positive operators which is uniformly norm-bounded is strongly convergent (to a positive operator).

Proof. For every $x \in H$, $(\langle A_n x, x \rangle)$ is an increasing sequence of positive numbers bounded above by $\sup ||A_n|| \cdot ||x||^2$, hence convergent. By the polarization formula it follows that $(\langle A_n x, y \rangle)$ is convergent for every $x, y \in H$. Hence A_n converges weakly to some $A \geqslant A_n$. But then

$$||(A - A_n)^{\frac{1}{2}} x||^2 = \langle (A - A_n)^{\frac{1}{2}} x, (A - A_n)^{\frac{1}{2}} x \rangle = \langle (A - A_n) x, x \rangle \to 0,$$

i.e. $(A - A_n)^{\frac{1}{2}} \to 0$ strongly and hence also $A_n \to A$ strongly. □

References

[BCH94] Paul Baum, Alain Connes, and Nigel Higson, *Classifying Space for Proper Actions and K-Theory of Group C*-algebras*, Contemp. Math. **167** (1994), 240–291.

[Bou63] N. Bourbaki, *Intégration – chapitres 7 & 8*, Hermann, 1963.

[Kas88] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201.

E-mail address: michael.walter@gmail.com