

A SHORT REMARK ON CONVERGENCE IN STOCHASTICS

1 Theorem. Let $(X_n), (Y_n), X, Y$ be random variables on the same probability space, and let $c \in \mathbb{R}$. Then:

- (i) $X_n \xrightarrow{\mathbb{P}} X, Y_n \xrightarrow{\mathbb{P}} Y \rightarrow X_n + Y_n \xrightarrow{\mathbb{P}} X + Y$ (same for almost sure convergence)
- (ii) $X_n \xrightarrow{\mathcal{P}} c \Rightarrow X_n \xrightarrow{\mathbb{P}} c$

2 Corollary. (i) cannot hold for convergence in distribution.

Proof. Otherwise we could lift convergence in distribution to convergence in probability:

$$\begin{aligned} X_n &\xrightarrow{\mathcal{P}} X \\ \stackrel{(i)}{\Rightarrow} X_n - X &\xrightarrow{\mathcal{P}} 0 \\ \stackrel{(ii)}{\Rightarrow} X_n - X &\xrightarrow{\mathbb{P}} 0 \\ \stackrel{(i)}{\Rightarrow} X_n &\xrightarrow{\mathbb{P}} X \end{aligned}$$

But we know that the former is strictly weaker than the latter. \square

3 Corollary. (ii) cannot apply to convergence in probability together with almost sure convergence.

Proof. By the same argument we could lift convergence in probability to almost sure convergence. Contradiction. \square