Mechanics*

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1 Generalized coordinates

Consider a physical system of N particles which is at each point in time fully specified by its Cartesian coordinates $\vec{r}_k \in \mathbb{R}^3$, $k = 1, \ldots, N$, that is, $3N$ parameters.

1 Definition. An m-dimensional *configuration space* is a subset Q of $\mathbb{R}^m$ together with mappings $\vec{f}_k : Q \times \mathbb{R} \to \mathbb{R}^3$ such that for each set of *possible* Cartesian coordinates $\vec{r}_k$ at time t we can find a *configuration* $\vec{q} \in Q$ satisfying

$$\vec{r}_k = \vec{f}_k(\vec{q}, t)$$

The components q_i of $\vec{q}$ are called *generalized coordinates.*

In the following we shall identify Cartesian coordinates $\vec{r}_k$ and the corresponding mappings $\vec{f}$ where appropriate.

2 Definition. The minimal dimension of configuration space is called the number of *degrees of freedom.*

Clearly, we can have no more than $3N$ degrees of freedom. Often, the number of degrees of freedom is less than $3N$, since the movement of particles in space is constrained in various ways.

3 Definition. A constraint is called *holonomic* if it can be expressed as

$$f(\vec{r}_1, \ldots, \vec{r}_n, t) = 0$$

for a function $f \in C^1(\mathbb{R}^{3N} \times \mathbb{R}, \mathbb{R})$ of the Cartesian coordinates and time.

A constraint is called *rhenomorous* if is explicitely time-dependent, and *scleronomous* otherwise.

4 Theorem. *Each independent holonomic constraint with a non-vanishing partial coordinate derivative reduces the number of degrees of freedom by one.*

Proof. Without loss of generality we assume that there is a single holonomic constraint which is expressed by

$$f(\vec{r}_1, \ldots, \vec{r}_n, t) = 0$$

By assumption f is not trivial and

$$\frac{\partial f}{\partial r_{i,j}} \neq 0$$

everywhere for a certain coordinate $r_{i,j}$. Thus the claim follows by the Implicit Function Theorem. □

The partial derivative condition is necessary as can be seen by inspection of the holonomic constraint $xy = 0$.

2 d'Alembert's Principle

Consider a physical system of N particles with positions $\vec{r}_k$ and L holonomic constraints.

5 Definition. An instantaneous ($t = const$) translation

$$\vec{r}_k \mapsto \vec{r}_k + \delta\vec{r}_k \quad (k = 1, \ldots, N)$$

satisfying the constraints of the system is called *virtual displacement.*

6 Definition. The force $\vec{F}_k$ acting on particle k can be split into an *external force* $\vec{F}^e_k$ and the *internal force* $\vec{F}^i_k$ maintaining the constraints:

$$\vec{F}_k = \vec{F}^e_k + \vec{F}^i_k$$

The internal force can be split further into the sum

$$\vec{F}^i_k = \sum_{l=1}^{L} \vec{F}^i_{l,k}$$

where each summand corresponds to a particular constraint l.

7 Theorem (d'Alembert's principle)**.** *If the internal forces corresponding to each holonomic constraint $f_l = 0$ act perpendicular to the hyperplane defined by the latter, that is, if*

$$\vec{F}^i_{l,k} \parallel \frac{\partial}{\partial \vec{r}_k} f_l(\vec{r}_1, \ldots, \vec{r}_N, t)$$

then the internal forces perform no virtual work*:*

$$\sum_{k=1}^{N} \left(\dot{\vec{p}}_k - \vec{F}^e_k \right) \cdot \delta\vec{r}_k = \sum_{k=1}^{N} \vec{F}^i_k \cdot \delta\vec{r}_k = 0$$

Proof. XXX □

We shall note that by its preconditions d'Alembert's principle is unable to model friction. Also we shall keep in mind that internal forces generally perform work (as opposed to *virtual* work); consider e.g. the system of a single particle being lifted by an elevator.

By above's theorem, given L constraints we can choose independent generalized coordinates $q_1, \ldots, q_{3N-L}$ so that the constraints are always satisfied.

8 Definition. We define the *generalized force* acting on coordinate q_i to be

$$Q_i := \sum_{k=1}^{N} \vec{F}^e_k \cdot \frac{\partial \vec{r}_k}{\partial q_i}$$

*inofficial notes loosely based on lectures by Prof. Nierste, Karlsruhe

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The following theorem shows that this definition is a useful generalization of the classical forces.

9 Theorem. *Under the conditions of d'Alembert's principle and given generalized coordinates q_i as stated above, the trajectory $\vec{q} : [t_1, t_2] \to \mathbb{R}^{3N-L}$ satisfies the following system of ODEs:*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, \ldots, 3N - L)$$

with the kinetic energy $T := \sum_{k=1}^{N} \frac{m_k}{2} \dot{\vec{r}}_k^{\,2}$.

Proof. By d'Alembert's principle:

$$\begin{aligned}
0 &= \sum_{k=1}^{N} \left(\dot{\vec{p}}_k - \vec{F}_k^e\right) \cdot \delta\vec{r}_k \\
&= \sum_{k=1}^{N} \left(\dot{\vec{p}}_k - \vec{F}_k^e\right) \cdot \sum_{i=1}^{3N-L} \frac{\partial \vec{r}_k}{\partial q_i} \delta q_i \\
&= \sum_{k=1}^{N} \sum_{i=1}^{3N-L} \dot{\vec{p}}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i} \delta q_i - \sum_{i=1}^{3N-L} Q_i \delta q_i \\
&= \sum_{i=1}^{3N-L} \left(\sum_{k=1}^{N} \dot{\vec{p}}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i} - Q_i \right) \delta q_i
\end{aligned}$$

By choice of coordinates the virtual displacements δq_i are independent and arbitrary, thus it follows that:

$$\begin{aligned}
Q_i &= \sum_{k=1}^{N} \dot{\vec{p}}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i} = \sum_{k=1}^{N} m \ddot{\vec{r}}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i} \\
&= \sum_{k=1}^{N} \frac{d}{dt} \left(m\dot{\vec{r}}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i} \right) - m\dot{\vec{r}}_k \frac{d}{dt}\frac{\partial \vec{r}_k}{\partial q_i} \\
&= \sum_{k=1}^{N} \frac{d}{dt} \left(m\dot{\vec{r}}_k \cdot \frac{\partial \dot{\vec{r}}_k}{\partial \dot{q}_i} \right) - m\dot{\vec{r}}_k \frac{\partial \dot{\vec{r}}_k}{\partial q_i} \\
&= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}
\end{aligned}$$

since $\frac{\partial \dot{\vec{r}}_k}{\partial \dot{q}_i} = \frac{\partial \vec{r}_k}{\partial q_i}$ by $\dot{\vec{r}}_k = \sum_{i=1}^{3N-L} \frac{\partial \vec{r}_k}{\partial q_i} \dot{q}_i + \frac{\partial \vec{r}_k}{\partial t}$. □

3 Lagrange's Equations

10 Definition. We now explore the situation in which each external force $\vec{F}$ can be derived from a *generalized potential* $U : \mathbb{R}^3 \to \mathbb{R}$ by

$$\vec{F} = -\frac{\partial U}{\partial \vec{r}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}}$$

(If a potential U exists but only depends on $\vec{r}$, that is, if $\frac{\partial U}{\partial \dot{\vec{r}}} = 0$, we say that the force $\vec{F}$ is *conservative*.)

If generalized potentials exist for each force, we can find a generalized potential $U : \mathbb{R}^{3N} \to R$ for the entire physical system such that

$$\vec{F}_k^e = -\frac{\partial U}{\partial \vec{r}_k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}_k} \quad (k = 1, \ldots, N)$$

In fact, we can choose $U := \sum_{k=1}^{N} U_k$ where U_k is the generalized potential corresponding to particle k.

11 Definition. Given a physical system with potential U, we define the *Lagrangian* as

$$\mathcal{L}(\vec{q}, \dot{\vec{q}}, t) := T(\vec{q}, \dot{\vec{q}}, t) - U(\vec{q}, \dot{\vec{q}}, t)$$

12 Theorem (Lagrange's equations, type 2)**.** *Consider a physical system with potential U and generalized coordinates satisfying the conditions of theorem 9. Then the trajectory $\vec{q} : [t_1, t_2] \to \mathbb{R}^{3N-L}$ satisfies the following system of ODEs:*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad (i = 1, \ldots, 3N - L)$$

Proof. We have

$$\begin{aligned}
Q_i &= \sum_{k=1}^{N} \vec{F}_k^e \cdot \frac{\partial \vec{r}_k}{\partial q_i} = \sum_{k=1}^{N} \left(-\frac{\partial U}{\partial \vec{r}_k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}_k} \right) \cdot \frac{\partial \vec{r}_k}{\partial q_i} \\
&= \sum_{k=1}^{N} -\frac{\partial U}{\partial \vec{r}_k} \cdot \frac{\partial \vec{r}_k}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}_k}\frac{\partial \dot{\vec{r}}_k}{\partial \dot{q}_i} \\
&= -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i}
\end{aligned}$$

Hence, by theorem 9:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i} \\
\Rightarrow &\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_i} - \frac{\partial (T-U)}{\partial q_i} = 0
\end{aligned}$$

and the claim follows. □

We now derive another kind of Lagrange's equations which can be useful (a) if it is hard to find independent generalized coordinates, (b) if one is interested in the inner forces, (c) in the case of non-holonomic constraints.

13 Definition. We say that a constraint

$$\sum_{i=1}^{J} a_i dq_i + a_0 dt = 0$$

(where $a_i \in \mathrm{C}(\mathbb{R}^J \times \mathbb{R})$ are functions of the generalized coordinates and time) is expressed *in differential form*.

14 Lemma. *A holonomic constraint*

$$f(q_1, \ldots, q_m) = 0$$

can be expressed in differential form by

$$\sum_{i=1}^{J} \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial t} dt = 0$$

Proof. The latter condition is clearly necessary, but it is also sufficient by the Fundamental Theorem of Calculus given initial values. □

15 Theorem (Lagrange's equation, type 1)**.** *Consider a physical system with J generalized coordinates, L differential constraints*

$$\sum_{i=1}^{J} a_{l,i} dq_i + a_{l,0} dt = 0 \quad (l = 1, \ldots, L)$$

and Lagrangian $\mathcal{L}$. Then there exist functions $\lambda_l \in \mathrm{C}(\mathbb{R}^J \times \mathbb{R})$ of the generalized coordinates and time such that the trajectory $\vec{q}: [t_1, t_2] \to \mathbb{R}^J$ satisfies the following system of ODEs:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{l=1}^{L} \lambda_l a_{l,i} \quad (i = 1, \ldots, J)$$

where the right-hand side is the generalized Q_i^i acting on coordinate i.

Proof. By a variation on the proof of theorem 9 we find that

$$\sum_{i=1}^{J} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \delta q_i = 0$$

but we cannot conclude that the individual factors vanish, since the generalized coordinates might not be independent. The virtual displacements satisfy

$$\sum_{i=1}^{J} a_{l,i} \delta q_i = 0 \quad (l = 1, \ldots, L)$$

(since $t = const$). It follows that for arbitrary $\lambda_l \in \mathrm{C}(\mathbb{R}^J \times \mathbb{R})$ we have

$$\begin{aligned} &\sum_{i,l} \lambda_l a_{l,i} \delta q_i = 0 \\ \Rightarrow &\sum_{i=1}^{J} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_{l}^{L} \lambda_l a_{l,i} \right) \delta q_i = 0 \end{aligned}$$

Without loss of generality we let $q_1, \ldots, q_{J-L}$ be independent coordinates, thus $\delta q_1, \ldots, \delta q_{J-L}$ are independent and $\delta q_{J-L+1}, \ldots, q_J$ are specified by the second-last equation. But this means that $(a_{l,i})$ ($l = 1, \ldots, L$, $i = J - L + 1, \ldots, J$) is an invertible $L \times L$ matrix, and it follows that we can choose λ_l such that for $i = J - L + 1, \ldots, J$ we have

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_{l}^{L} \lambda_l a_{l,i} = 0$$

But then we notice that

$$\sum_{i=1}^{J-L} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_{l}^{L} \lambda_l a_{l,i} \right) \delta q_i = 0$$

and by the independence of $q_1, \ldots, q_{J-L}$ we conclude that for all $i = 1, \ldots, J$ we have

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_{l}^{L} \lambda_l a_{l,i} = 0$$

from which the main claim follows. We further recognize the generalized inner forces by comparison with theorems 9 and 12. □

The λ_l are called *Lagrange multipliers.*

16 Corollary. *Given holonomic constraints $f_l = 0$ only, we can also choose the Lagrangian*

$$\mathcal{L}' := \mathcal{L} + \sum_{i,l} \lambda_l f_l(q_i, t)$$

Then the trajectory satisfies

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}'}{\partial q_i} = 0$$

and the generalized inner force $Q_{l,i}^i$ (corresponding to constraint l acting on coordinate i) is given by $\lambda_l \frac{\partial f_l}{\partial q_i}$.

Proof. By preceding lemma we find that

$$\begin{aligned} &\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{l=1}^{L} \lambda_l a_{l,i} \\ \Leftrightarrow &\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{l=1}^{L} \lambda_l \frac{\partial f_l}{\partial q_i} \\ \Leftrightarrow &\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial}{\partial q_i} \sum_{l=1}^{L} \lambda_l f_l \\ \Leftrightarrow &\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}'}{\partial q_i} = 0 \end{aligned}$$

thus follows the claim by theorem 15. The second claim follows from the uniqueness of basis decomposition, since

$$\begin{aligned} Q_{l,i}^i = &\sum_k \vec{F}_{l,k}^i \cdot \frac{\partial \vec{r}_k}{\partial q_i} = \sum_k \mu_k \frac{\partial f_l}{\partial \vec{r}_k} \cdot \frac{\partial \vec{r}_k}{\partial q_i} \\ = &\mu \frac{\partial f_l}{\partial q_i} \end{aligned}$$

and

$$\sum_{l=1}^{L} \lambda_l \frac{\partial f_l}{\partial q_i} = Q_i^i = \sum_{l=1}^{L} Q_{l,i}^i$$

□

4 Hamilton's Principle

17 Definition. The functional

$$S[\vec{q}] := \int_{t_1}^{t_2} L(\vec{q}(t), \dot{\vec{q}}(t), t) dt$$

is called *action.*

18 Theorem (Hamilton's principle)**.** *Consider a physical system and generalized coordinates for which theorem 12 applies. Then the action is stationary for the actual trajectory $\vec{q}: [t_1, t_2] \to \mathbb{R}^{3N-L}$ the system undergoes.*

Proof. By the calculus of variations, S is stationary if and only if the Lagrange Equations of type 2 are satisfied. □

19 Corollary. *Consider a physical system and generalized coordinates for which theorem 12 applies. Then d'Alambert's principle and Hamilton's principle are equivalent.*

This statement makes most sense when one of both principles is taken as an axiom.

20 Theorem. *The Lagrangian is not unique. Particulary, time derivatives of the form $\frac{d}{dt}f(\vec{q}(t), t)$ can be added to it without changing the trajectory.*

Proof.

$$\begin{aligned} S' &= \int_{t_1}^{t_2} L(\vec{q}(t), \dot{\vec{q}}(t), t) + \frac{d}{dt} f(\vec{q}(t), t) dt \\ &= \int_{t_1}^{t_2} L(\vec{q}(t), \dot{\vec{q}}(t), t) dt + \int_{t_1}^{t_2} \frac{d}{dt} f(\vec{q}(t), t) dt \\ &= S + f(\vec{q}(t_2), t_2) - f(\vec{q}(t_1), t_1) \end{aligned}$$

We observe that S' is stationary if and only if S is stationary, which proves the claim. □

5 Conservation Laws

21 Definition. The *generalized momentum* (or *conjugate momentum*) corresponding to the generalized coordinate q_i given the Lagrangian $\mathcal{L}$ is defined as

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

We say that a coordinate q_i is *cyclic* if

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

22 Theorem. *Consider a physical system and generalized coordinates for which Lagrange's equations apply. Then q_i is cyclic if and only if the corresponding generalized momentum p_i is conserved; that is:*

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \Leftrightarrow \quad p_i = const.$$

Proof. The claim directly follows from theorem 12. □

If x is a Cartesian coordinate, then p_x represents the corresponding Cartesian momentum coordinate Π_x.

If θ is an angle around an axis $\vec{n}$, then p_θ represents the angular momentum coordinate corresponding to that axis, $L \cdot \vec{n}$.

23 Definition. The *Hamiltonian* is defined as

$$H(\vec{q}, \dot{\vec{q}}, t) := \sum_{i=1}^{3N-L} p_i \dot{q}_i \ - \ \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$$

24 Lemma. *Given the Lagrangian $\mathcal{L} = T - U$ from theorem 12 and time-independent coordinates, we find that*

$$H = T + U - \sum_{i=1}^{3N-L} \frac{\partial U}{\partial \dot{q}_i} \dot{q}_i$$

Proof. Since

$$T = \sum_{k=1}^{N} \frac{m_k}{2} \dot{\vec{r}}_k^2 = \sum_{k=1}^{N} \frac{m_k}{2} \left(\sum_{i=1}^{3N-L} \frac{\partial \vec{r}}{\partial q_i} \dot{q}_i \ + \ \underbrace{\frac{\partial \vec{r}}{\partial t}}_{=0} \right)^2$$

it follows that

$$\sum_{i=1}^{3N-L} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = 2T$$

hence

$$\begin{aligned} H = \sum_{i=1}^{3N-L} p_i \dot{q}_i \ - \ \mathcal{L} &= \sum_{i=1}^{3N-L} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \ - \ \mathcal{L} \\ &= 2T - \mathcal{L} - \sum_{i=1}^{3N-L} \frac{\partial U}{\partial \dot{q}_i} \dot{q}_i \end{aligned}$$

by which the claim follows immediately. □

If U is velocity-independent, the Hamiltonian reduces to the total energy.

25 Theorem. *Consider a physical system and generalized coordinates for which theorem 12 applies. Then we have:*

$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

and hence the Hamiltonian is conserved if and only if the Lagrangian is not explicitely time-dependent.

Proof. By Lagrange's equations we see that

$$\begin{aligned} -\frac{\partial \mathcal{L}}{\partial t} &= \sum_{i=1}^{3N-L} \left(\frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) - \frac{d\mathcal{L}}{dt} \\ &= \sum_{i=1}^{3N-L} \left(\left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) - \frac{d\mathcal{L}}{dt} \\ &= \frac{d}{dt} \left(\sum_{i=1}^{3N-L} p_i \dot{q}_i \right) - \frac{d\mathcal{L}}{dt} = \frac{dH}{dt} \end{aligned}$$

which shows the claim. □

26 Theorem (Noether)**.** *Let*

$$\begin{aligned} q_i \mapsto q_i' &= f_i(\vec{q}, \dot{\vec{q}}, t, \epsilon) \\ t \mapsto t' &= f_0(\vec{q}, \dot{\vec{q}}, t, \epsilon) \end{aligned}$$

be a coordinate transformation parametrized by ϵ such that $f_i(\varphi = 0) = q_i$, $f_0(\varphi = 0) = t$. If

$$\mathcal{L}(\vec{q}\,', \dot{\vec{q}}\,', t') dt' = \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) dt + \epsilon dF + O(\epsilon^2)$$

for a function $F(\vec{q}, t)$, then the Noether charge

$$Q = \sum_{i=1}^{3N-L} p_i \frac{\partial f_i}{\partial \epsilon} \ - \ H \frac{\partial f_0}{\partial \epsilon} \ - \ F$$

is conserved.

Proof. XXX □

27 Example (Homogeneity of time)**.** If Noether's theorem applies to $t \mapsto t + \epsilon$, then the Hamiltonian H is conserved.

28 Example (Homogeneity of space)**.** If Noether's theorem applies to $\vec{r} \mapsto \vec{r} + \epsilon \vec{e}_i$, then the ith Cartesian momentum coordinate is conserved.

29 Example (Isotropy of space)**.** If Noether's theorem applies to

$$\vec{r} \mapsto \vec{r} + \epsilon\,(\vec{n} \times \vec{r}) \quad (||n|| = 1)$$

then the angular momentum coordinate corresponding to $\vec{n}$ is conserved.

30 Example (Galilei transformation)**.** Consider a system of N particles transformed by the same Galilei transformation

$$\vec{r}_k \mapsto \vec{r}_k + \epsilon\vec{v}t \quad (||v|| = 1, k = 1, \ldots, N)$$

Given the potential only depends on relative distances between particles, that is if

$$\mathcal{L} = \sum_{k=1}^{N} \frac{m_k}{2}\dot{\vec{r}}_k^2 \; - \; \sum_{k \neq l} U(\underbrace{\vec{r}_k - \vec{r}_l}_{=const})$$

we find that

$$\begin{aligned}&\mathcal{L}(\vec{r}_k + \epsilon\vec{v}t, \dot{\vec{r}}_k + \epsilon\vec{v}, t)\\ =&\mathcal{L}(\vec{r}_k, \dot{\vec{r}}_k, t) + \epsilon\sum_{k=1}^{N} m_k\dot{\vec{r}}_k \cdot \vec{v} + O(\epsilon^2)\end{aligned}$$

hence we can apply Noether's theorem with

$$F = \sum_{k=1}^{N} m_k\vec{r}_k \cdot \vec{v} = M\vec{r}_S \cdot \vec{v}$$

resulting in the conserved quantity

$$\begin{aligned}Q =& \sum_{k=1}^{N} \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_k} \cdot \vec{v}t - M\vec{r}_S \cdot \vec{v}\\ =& \sum_{k=1}^{N} \vec{P}i_k \cdot \vec{v}t - M\vec{r}_S \cdot \vec{v}\\ =& \left(\vec{P}t - M\vec{r}_s\right) \cdot \vec{v}\end{aligned}$$

where r_s is the center of mass and P is the total momentum of the system (see next chapter).

6 Rigid Bodies

31 Definition. A *rigid body* is a body that does not change over time; that is, for points $\vec{r}_k$, $\vec{r}_l$ we demand that

$$||\vec{r}_k - \vec{r}_l|| = const.$$

32 Definition. The *center of mass* is defined as

$$\vec{r}_S := \sum_{k=1}^{N} m_k\vec{r}_k$$

33 Theorem. *Let O be an arbitrary point attached to the rigid body. Then the motion of each particle of a rigid body is composed of a translation and and a rotation around O, that is:*

$$\dot{\vec{r}}_k = \vec{v}_O + \vec{\omega} \times \vec{r}_k'$$

where $\vec{r}_k$ is the kth particle's position in the inertial system, $\vec{v}_O$ is the velocity of the origin in the inertial system and $\vec{r}_k'$ is the particle's position relative to the origin.

Proof. This is a consequence of Euler's theorem. □

Note that the origin can be chosen arbitrarily, as long as it is attached to the rigid body.

34 Theorem. *In a coordinate system* attached to the rigid body *we can relate angular momentum and angular velocity as follows:*

$$\vec{L} = \Theta\vec{\omega}$$

where Θ is the inertia tensor *defined by*

$$\Theta_{i,j} := \sum_{k=1}^{N} m_k \left(\vec{r}_k^2\delta_{i,j} - r_{k,i}r_{k,j}\right)$$

Proof. Since the given coordinate system is attached to the rigid body, the particles are solely rotating around the frame's origin:

$$\dot{\vec{r}}_k = \vec{\omega} \times \vec{r}_k$$

we have

$$\begin{aligned}\vec{L} =& \sum_{k=1}^{N} m_k\left(\vec{r}_k \times \dot{\vec{r}}_k\right) = \sum_{k=1}^{N} m_k\left(\vec{r}_k \times (\vec{\omega} \times \vec{r}_k)\right)\\ =& \sum_{k=1}^{N} m_k\left(\vec{r}_k^2\vec{\omega} - (\vec{r}_k \cdot \vec{\omega})\,\vec{r}_k\right)\\ =& \sum_{j=1}^{3}\sum_{k=1}^{N} m_k\left(\vec{r}_k^2\vec{\omega} - \vec{r}_{k,j}\vec{r}_k\omega_j\right)\end{aligned}$$

(using the Grassman identity), i.e.

$$\begin{aligned}L_i =& \sum_{j=1}^{3}\sum_{k=1}^{N} m_k\left(\vec{r}_k^2\omega_i - \vec{r}_{k,j}\vec{r}_{k,i}\omega_j\right)\\ =& \sum_{j=1}^{3}\sum_{k=1}^{N} m_k\left(\vec{r}_k^2\delta_{i,j}\omega_j - \vec{r}_{k,i}\vec{r}_{k,j}\omega_j\right)\\ =& \sum_{j=1}^{3}\left(\sum_{k=1}^{N} m_k\left(\vec{r}_k^2\delta_{i,j} - \vec{r}_{k,i}\vec{r}_{k,j}\right)\right)\omega_j\end{aligned}$$

from which we recognize matrix multiplication with Θ. □

We note that both $\vec{L}$ and Θ depend on the choice of origin. We also appreciate that the preceding theorem parallels the familiar relationship $\vec{\Pi} = m\vec{v}$ between momentum and and velocity.

35 Definition. We define the *moment of inertia* about an axis $\vec{n}$ ($||\vec{n}|| = 1$) to be

$$\Theta_{\vec{n}} = \vec{n}^T\Theta\vec{n}$$

36 Theorem. *Θ is orthogonally diagonalizable; that is, there exists an $O \in O(3)$ such that*

$$O^T\Theta O = \begin{pmatrix}\Theta_1 & & \\ & \Theta_2 & \\ & & \Theta_3\end{pmatrix}$$

The eigenvalues Θ_i are called principal moments of inertia*, the corresponding coordinate axes (that is, the columns of O) are called* principal axes.

Proof. This follows from a basic theorem in linear algebra. □

37 Theorem. *Given a point O attached to the rigid body, the total kinetic energy of the rigid body is*

$$T = \frac{M}{2}\vec{v}_O^2 \ + \ (\vec{v}_O \times \vec{\omega}) \cdot \vec{r}_S' \ + \ \frac{1}{2}\vec{\omega}^T \Theta \vec{\omega}$$

where $\vec{v}_O$ is the velocity of the origin, and the center of mass $\vec{r}_S'$, Θ and $\vec{\omega}$ are taken in a coordinate system with origin O.

Proof. By theorem 33 we have

$$\begin{aligned}
T =& \sum_{k=1}^{N} \frac{m_k}{2}\dot{\vec{r}}_k^2 = \sum_{k=1}^{N} \frac{m_k}{2}\left(\vec{v}_O + \vec{\omega} \times \vec{r}_k'\right)^2 \\
=& \sum_{k=1}^{N} \frac{m_k}{2}\left(\vec{v}_O^2 + 2\vec{v}_O \cdot (\vec{\omega} \times \vec{r}_k') + (\vec{\omega} \times \vec{r}_k')^2\right) \\
=& \frac{M}{2}\vec{v}_O^2 + \sum_{k=1}^{N} m_k \vec{v}_O \cdot (\vec{\omega} \times \vec{r}_k') \\
&+ \sum_{k=1}^{N} \frac{m_k}{2}\dot{\vec{r}}_k' \cdot (\vec{\omega} \times \vec{r}_k')
\end{aligned}$$

$$\begin{aligned}
=& \frac{M}{2}\vec{v}_O^2 + (\vec{v}_O \times \vec{\omega}) \cdot \sum_{k=1}^{N} m_k \vec{r}_k' \\
&+ \sum_{k=1}^{N} \frac{m_k}{2}\vec{\omega} \cdot \left(\vec{r}_k' \times \dot{\vec{r}}_k'\right) \\
=& \frac{M}{2}\vec{v}_O^2 + (\vec{v}_O \times \vec{\omega}) \cdot \vec{r}_S' + \sum_{k=1}^{N} \frac{1}{2}\vec{\omega}^T \vec{L} \\
=& \frac{M}{2}\vec{v}_O^2 + (\vec{v}_O \times \vec{\omega}) \cdot \vec{r}_S' + \sum_{k=1}^{N} \frac{1}{2}\vec{\omega}^T \Theta \vec{\omega}
\end{aligned}$$

by $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$ and theorem 34. □

We note the analogy between $T_{trans} = \frac{1}{2}\vec{v}^T\vec{p} = \frac{1}{2}\vec{v}^T m \vec{v}$ and $T_{rot} = \frac{1}{2}\vec{\omega}^T L = \frac{1}{2}\vec{\omega}^T \Theta \vec{\omega}$.

38 Corollary. *If a rigid body is fixed at a given point O, we have*

$$T = \frac{1}{2}\vec{\omega}^T \Theta \vec{\omega}$$

where Θ, $\vec{\omega}$ are taken in a coordinate system with origin O.

Proof. We apply the theorem and note that $\vec{v}_O = 0$. □

39 Corollary.

$$T = \frac{M}{2}\dot{\vec{r}}_S^2 \ + \ \frac{1}{2}\vec{\omega}^T \Theta \vec{\omega}$$

where Θ, $\vec{\omega}$ are taken in a coordinate system with the center of mass as the origin.

Proof. We apply the theorem and note that $\vec{r}_S' = 0$ in the given coordinate system. □

By theorem 36 and the corollaries we can often find a simple kinetic energy term. We state the following theorem without proof:

40 Theorem. *By taking the limit $N \to \infty$ we arrive at*

$$\begin{aligned}
\vec{r}_S &= \int_V \rho(\vec{r})\vec{r}_k dV \\
\Theta_{i,j} &= \int_V \rho(\vec{r}) \left(\vec{r}^2 \delta_{i,j} - r_i r_j\right) dV
\end{aligned}$$

for a continous mass distribution with density ρ.

41 Theorem (Steiner)**.** *If Θ_S is the moment of inertia about a given axis $\vec{n}$ ($||\vec{n}|| = 1$) through the center of mass, then*

$$\Theta_{\vec{n}} := \Theta_{S;\vec{n}} + Ma^2$$

is the moment of inertia about any parallel axis with distance a.

Proof. By the preceding theorem, the inertia tensor with respect to a coordinate system with the center of mass as the origin is given by

$$\Theta_{S;i,j} = \int_V \rho(\vec{r}) \left(\vec{r}^2 \delta_{i,j} - r_i r_j\right) dV$$

and with respect to a coordinate system translated into a point with coordinates $\vec{b}$ by

$$\begin{aligned}
&\Theta_{i,j} \\
&= \int_V \rho(\vec{r}) \left((\vec{r} - \vec{b})^2 \delta_{i,j} - (r_i - b_i)(r_j - b_j)\right) dV \\
&= \Theta_{S;i,j} + \int_V \rho(\vec{r}) \left(\vec{b}^2 \delta_{i,j} - b_i b_j\right) dV \\
&= \Theta_{S;i,j} + M \left(\vec{b}^2 \delta_{i,j} - b_i b_j\right)
\end{aligned}$$

since the linear terms vanish by choice of the center of mass as the origin. Hence it follows that

$$\begin{aligned}
\Theta_{\vec{n}} =& \vec{n}^T \Theta \vec{n} = \vec{n}^T \Theta_S \vec{n} + M \vec{n}^T \left(\vec{b}^2 I - \vec{b}\vec{b}^T\right) \vec{n} \\
=& \vec{n}^T \Theta_S \vec{n} + M \underbrace{\left(\vec{b}^2 - \left(\vec{b} \cdot \vec{n}\right)^2\right)}_{=a^2}
\end{aligned}$$

□

42 Theorem (Euler's equations)**.** *In a coordinate system where the inertia tensor is diagonal the angular velocity satisfies the following system of ODEs:*

$$\begin{aligned}
\Theta_1 \dot{\omega}_1 + (\Theta_3 - \Theta_2)\, \omega_2 \omega_3 &= M_1 \\
\Theta_2 \dot{\omega}_2 + (\Theta_1 - \Theta_3)\, \omega_1 \omega_3 &= M_2 \\
\Theta_3 \dot{\omega}_3 + (\Theta_2 - \Theta_1)\, \omega_1 \omega_2 &= M_3
\end{aligned}$$

where $\vec{M}$ is the torque.

Proof. Since

$$\frac{d}{dt}|_{inertial} = \frac{d}{dt}|_{body} + \vec{\omega}\times$$

we conclude that

$$\vec{M} = \frac{d}{dt}(\Theta\vec{\omega}) + \vec{\omega} \times (\Theta\vec{\omega})$$

□

6.1 Spinning Tops

43 Definition. A rotating rigid body is called a *spinning top.*

44 Lemma. *For constant rotational energy, the equation*

$$\frac{1}{2}\vec{\omega}^T \Theta \vec{\omega} = T_{rot}$$

describes an ellipsoid in angular velocity space, called the inertia ellipsoid.

45 Definition. A spinning top with two identical principal moments of inertia is called a *symmetrical top.* If all three principal moments of inertia are equal, it is called a *spherical top.*

Given a symmetrical top with $\Theta_1 = \Theta_2 = \Theta$, the inertia ellipsoid is *oblate* if $\Theta_3 < \Theta$ and *prolate* if $\Theta_3 > \Theta$.

46 Theorem. *Consider a spinning top without external forces. Then rotation around a principal axis is stable if and only if either the spinning top is spherical or the corresponding principal moment of inertia is the (strictly) largest or smallest.*

If it is stable, then small perturbation leads to precession *aboud the stable solution, that is, the axis of rotation moves around the principal axis.*

Proof. Without loss of generality we consider a rotation around the 3rd principal axis. Clearly

$$\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = const.$$

is a solution of Euler's equations ($M = 0!$). After small perturbation the angular velocity changes to

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad \omega_1, \omega_2 \ll \omega_3$$

Euler's equations are

$$\begin{aligned}
\Theta_1\dot{\omega}_1 + (\Theta_3 - \Theta_2)\omega_2\omega_3 &= 0 \\
\Theta_2\dot{\omega}_2 + (\Theta_1 - \Theta_3)\omega_1\omega_3 &= 0 \\
\Theta_3\dot{\omega}_3 + (\Theta_2 - \Theta_1)\omega_1\omega_2 &= 0
\end{aligned}$$

and clearly the solution is stable in case of a spherical top. Otherwise, we see by substituting into the third equation that

$$\dot{\omega}_3 = O(\frac{\omega_{1/2}}{\omega_3})$$

Thus ω_3 is constant to order $\frac{\omega_{1/2}}{\omega_3}$! By this approximation

$$\omega_2 = \frac{\Theta_1}{\Theta_2 - \Theta_3}\frac{\dot{\omega}_1}{\omega_3} \quad \Rightarrow \quad \dot{\omega}_2 = \frac{\Theta_1}{\Theta_2 - \Theta_3}\frac{\ddot{\omega}_1}{\omega_3}$$

which we plug into the second of Euler's equations resulting in

$$\ddot{\omega}_1 + \frac{(\Theta_1 - \Theta_3)(\Theta_2 - \Theta_3)}{\Theta_1\Theta_2}\omega_3^2\omega_1 = 0$$

By consideration of the solution space of this linear equation we conclude that ω_1 oscillates around 0 if and only if

$$\begin{aligned}
&\Theta_1 - \Theta_3)(\Theta_2 - \Theta_3) > 0 \\
\Leftrightarrow &\Theta_3 > \Theta_1, \Theta_2 \quad \text{or} \quad \Theta_3 < \Theta_1, \Theta_2
\end{aligned}$$

(and otherwise diverges). By symmetry, the same condition applies ω_2, Thus the claim follows. □

47 Theorem. *Given a symmetrical top, rotations are always stable and lead to precession about the axis of symmetry.*

Proof. Euler's equations for a symmetrical top with $\Theta_1 = \Theta_2 = \Theta$ are given by

$$\begin{aligned}
\Theta\dot{\omega}_1 + (\Theta_3 - \Theta)\omega_2\omega_3 &= 0 \\
\Theta\dot{\omega}_2 + (\Theta - \Theta_3)\omega_1\omega_3 &= 0 \\
\Theta_3\dot{\omega}_3 &= 0
\end{aligned}$$

Hence $\omega_3 = const$. By setting $\omega := \omega_1 + i\omega_2$ we can unify the first two ODEs into

$$\begin{aligned}
\dot{\omega} + \frac{\Theta - \Theta_3}{\Theta}\omega_3 i\omega &= 0 \\
\Rightarrow \frac{d\omega}{\omega} &= \frac{\Theta_3 - \Theta}{\Theta}\omega_3 i dt
\end{aligned}$$

The solution of this ODE is

$$\omega = C\exp(i\Omega t)$$

with $\Omega := \frac{\Theta_3 - \Theta}{\Theta}\omega_3 \in \mathbb{R}$. Thus ω moves on a circle around the origin, resulting in uniform rotation of the axis of rotation around the axis of symmetry of the spinning top. □

48 Definition. If after small perturbation the axis of rotation precesses around an axis, the "nodding" of the former with regards to the latter is called *nutation.*

6.2 Euler Angles

49 Theorem (Euler Angles)**.** *Every rotation $O \in SO(3)$ can be represented by the composition of three 2D rotation matrices as in*

$$\begin{aligned}
O = &\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \\
&\begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
=: &O_\varphi O_\theta O_\psi
\end{aligned}$$

The angles φ, θ, ψ are the Euler angles *corresponding to that rotation.*

If we use Euler angles to specify the orientation of a rigid body, O specifies the transformation from body-fixed to inertial axes. We call φ the precession angle *and θ the* nutation angle*. ψ specifies the rotation around the rigid body's z-axis.*

Proof. This is a basic theorem in linear algebra. □

50 Theorem. *If we use Euler angles to specify the orientation of a rigid body, then we can express angular velocity* in the thus defined coordinate system *by*

$$\vec{\omega} = \begin{pmatrix} \sin(\psi)\sin(\theta)\dot{\varphi} + \cos(\psi)\dot{\theta} \\ \cos(\psi)\sin(\theta)\dot{\varphi} - \sin(\psi)\dot{\theta} \\ \cos(\theta)\dot{\varphi} + \dot{\psi} \end{pmatrix}$$

Proof. ψ rotates around the rigid body's z-axis, hence

$$\omega_\psi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\psi}$$

θ rotates around the intermediate x-axis given by $O_\varphi O_\theta \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, so

$$\omega_\theta = O_\psi^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\theta} = \begin{pmatrix} \cos(\psi) \\ -\sin(\psi) \\ 0 \end{pmatrix} \dot{\theta}$$

Finally, φ rotates around the inertial z-axis, thus

$$\begin{aligned} \omega_\varphi =& O^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\varphi} = O_\psi^T \begin{pmatrix} 0 \\ \sin(\theta) \\ \cos(\theta) \end{pmatrix} \dot{\varphi} \\ =& \begin{pmatrix} \sin(\psi)\sin(\theta) \\ \cos(\psi)\sin(\theta) \\ \cos(\theta) \end{pmatrix} \dot{\varphi} \end{aligned}$$

Angular velocity is a (pseudo) vector, thus we can add up the individual projections and the claim follows. □

7 Hamiltonian Mechanics

51 Definition. The *Hamiltonian* is defined as

$$H(\vec{p}, \vec{q}, t) := \sum_{i=1}^{3N-L} p_i \dot{q}_i \; - \; \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$$

i.e. as a function of the *canonical coordinates* $\vec{p}$, $\vec{q}$, t.

Contrast this definition and the one in chapter 4 which unfortunately differs.

Since the Hamiltonian just defined is a function of the canonical coordinates, we need to express the right-hand side in terms of these.

52 Theorem (Hamilton's equations)**.** *Lagrange's equations are equivalent to* Hamilton's equations*:*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Furthermore we have

$$\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = \frac{dH}{dt}$$

Proof. By using $p_i = \frac{\partial \mathcal{L}}{\partial q_i}$ and Lagrange's equation we find that

$$\begin{aligned} dH =& \sum_{i=1}^{3N-L} dp_i \dot{q}_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \\ & - \frac{\partial \mathcal{L}}{\partial t} dt \\ =& \sum_{i=1}^{3N-L} dp_i \dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i \; - \; \frac{\partial \mathcal{L}}{\partial t} dt \\ =& \sum_{i=1}^{3N-L} dp_i \dot{q}_i - \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) dq_i \; - \; \frac{\partial \mathcal{L}}{\partial t} dt \\ =& \sum_{i=1}^{3N-L} \dot{q}_i dp_i - \dot{p}_i dq_i \; - \; \frac{\partial \mathcal{L}}{\partial t} dt \end{aligned}$$

On the other hand we have

$$dH = \sum_{i=1}^{3N-L} \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \; + \; \frac{\partial H}{\partial t} dt$$

thus follow the claims by partial differentiation and consideration of theorem 25. □

53 Example (Central potential)**.** We recall that motion in a central potential happens in a plane, thus we use polar coordinates r, φ. We recall that

$$\mathcal{L} = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) - U(r)$$

The conjugate momenta are

$$\begin{aligned} p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \\ p_\varphi &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2 \dot{\varphi} \end{aligned}$$

It follows that the Hamiltonian is given by

$$\begin{aligned} H =& p_r \dot{r} + p_\varphi \dot{\varphi} - \mathcal{L} \\ =& \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) + U(r) \\ =& \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + U(r) \end{aligned}$$

(which could have also been calculated using the lemma in chapter 5). Thus Hamilton's equations are given by

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2} \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{mr^3} - U'(r) \\ \dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi} = 0 \end{aligned}$$

We notice that both H and $p_\varphi = L_z$ are conserved.

8 Small oscillations

54 Theorem. *Given a physical system with Hamiltonian $H = T + U$, time-independent potential and J time-independent coordinates.*

Then sufficiently small perturbations around local potential minima result in small harmonic oscillations, so-called normal modes*.*

Proof. Without loss of generality assume that the potential has a local minimum $U_{\min} = 0$ for $\vec{q} = 0$. Thus the Taylor expansion is given by

$$U = \frac{1}{2} \sum_{i,j} \underbrace{\frac{\partial^2 U}{\partial q_i \partial q_j}|_{\vec{q}=0}}_{=:u_{i,j}} q_i q_j + O(q_k^3)$$

(Linear terms vanish since we are expanding around the local minimum!) The kinetic energy is quadratic in $\dot{q}_i$, hence quadratic in $p_i = \frac{\partial T}{\partial \dot{q}_i}$, hence

$$T = \frac{1}{2} \sum_{i,j} \tilde{t}_{i,j} p_i p_j$$

In general, the $\tilde{t}_{i,j}$ are functions of the generalized coordinates. We expand

$$\tilde{t}_{i,j} = t_{i,j} + O(q_k)$$
$$\Rightarrow T = \frac{1}{2} \sum_{i,j} t_{i,j} p_i p_j + O(q_k p_i p_j)$$

and assuming that both generalized coordinates and momenta are small we neglect higher-order terms resulting in

$$H = T + U = \frac{1}{2} \sum_{i,j} \left(t_{i,j} p_i p_j + u_{i,j} q_i q_j \right)$$

By Hamilton's equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{1}{2} \sum_{j,k} u_{j,k} \left(\delta_{j,i} q_k + q_j \delta_{k,i} \right)$$
$$= -\sum_j u_{i,j} q_j$$
$$\dot{q}_i = \ldots = \sum_j t_{i,j} p_j$$

hence

$$\ddot{q}_i = \sum_j t_{i,j} \dot{p}_j = -\sum_{j,k} t_{i,j} u_{j,k} q_k$$

With the matrices $\hat{T} := (t_{i,j}, \hat{U} := (u_{i,j})$ an equivalent ODE is given by

$$\ddot{\vec{q}} = -\hat{T}\hat{U}\vec{q}$$

We attempt $\vec{q}(t) := \vec{q}_0 \exp(\pm i\omega t)$ which leads to

$$\left(\hat{T}\hat{U} - \omega^2 I_n \right) \vec{q}_0 = 0$$

"Physical" solutions of this equation require that all eigenvalues ω^2 be real and positive. Thus we consider the eigenvalue problem

$$\hat{T}\hat{U}\vec{a}_i = \lambda_i \vec{a}_i$$

Since $T = \frac{1}{2}\vec{p}^T \hat{T} \vec{p}$ is positive definite we find that $\hat{T}$ is orthogonally diagonalizable with positive eigenvalues:

$$O^T \hat{T} O = \operatorname{diag}(t_1, \ldots, t_n)$$

Consequently,

$$\sqrt{\hat{T}} := O \operatorname{diag}(\sqrt{t_1}, \ldots, \sqrt{t_n})$$

is positive definite with $\sqrt{\hat{T}}^2 = \hat{T}$. We define $\hat{U}' := \sqrt{\hat{T}}\hat{U}\sqrt{\hat{T}}$ and $a_i =: \sqrt{\hat{T}} a_i'$, and the eigenvalue problem reduces to

$$\hat{U}' a_i' = \lambda_i a_i'$$

But clearly U_i' is symmetric, thus all eigenvalues λ_i are real! And since U is positive definite by the minimum condition it follows that

$$0 < \vec{a}^T U \vec{a} = \vec{a}'^T U' \vec{a}'$$

for all $\vec{a} = \sqrt{\hat{T}}\vec{a}'$, hence U' is positive definite and all eigenvalues are positive, i.e. $0 < \omega^2 \in \mathbb{R}$.

We eventually note that solution have the form $\vec{q}(t) = \vec{q}_0 \sin(\omega t + \varphi)$, thus

$$\dot{\vec{p}} = -\hat{V}\vec{q} = -\hat{V}\vec{q}_0 \sin(\omega t + \varphi)$$
$$\Rightarrow \vec{p} = \frac{1}{\omega}\hat{V}\vec{q}_0 \cos(\omega t + \varphi) =: \vec{p}_0 \cos(\omega t + \varphi)$$

which is in agreement to our initial assumptation that small perturbations correspond to small generalized momenta (unless frequencies becomes too large!). □

9 Phase space

It follows from Hamilton's equations that, at any point in time, a physical system with J degrees of freedom is fully specified by J generalized coordinates and momenta.

55 Definition. We can interprete a *configuration* $\left(p_1 \ldots p_J q_1 \ldots q_J\right)^T$ as vectors in *phase space* $\mathbb{R}^{2J}$.

56 Lemma. *Given the configuration at $t = t_0$, the system's trajectory in phase space is determined for all time.*

Hence, two trajectories in phase space cannot cross or touch.

57 Example (Harmonic Oscillator)**.** Consider the harmonic oscillator with a single degree of freedom given by

$$\mathcal{L} = \frac{m}{2}\dot{q}^2 - \frac{m}{2}\omega^2 q^2$$
$$\Rightarrow H = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 q^2 \right)$$

The solution of Hamilton's equations with initial conditions $p(0) = p_0, q(0) = q_0$ is given by

$$p(t) = p_0 \cos(\omega t) - m\omega q_0 \sin(\omega t)$$
$$q(t) = q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t)$$

This describes a circle in phase space.

In what follows we give an exposure to *statistical mechanics*, the goal of which is to relate *macroscopic quantities* such as energy, pressure and volume to *microscopic physics* given by the Hamiltonian.

58 Definition. Consider a large number of physical systems with J degrees of freedom each. Then we define a density $\rho(\vec{p},\vec{q},t)$ called *phase space distribution* such that

$$dn = \rho(\vec{p},\vec{q},t)d\omega$$

where dn is the number of systems per phase space volume elemnt $d\omega := d\vec{p}d\vec{q} = dp_1 \ldots dp_J dq_1 \ldots dq_J$.

59 Theorem (Liouville)**.**

$$\frac{d\rho}{dt} = 0$$

Proof. We first define

$$\vec{v} = \left(\dot{p}_1 \ldots \dot{p}_J \dot{q}_1 \ldots \dot{q}_J\right)$$

Consider a fixed phase space volume element ω. Since particles leaving ω must do so through the surface $\partial\omega$, we find that

$$-\frac{d\omega}{dt} = -\frac{\partial}{\partial t}\int_\omega \rho d\omega = \int_{\partial\omega} \rho\vec{v}\cdot\vec{n}dS = \int_\omega \vec{\nabla}\cdot(\rho\vec{v})\, d\omega$$

by Gauss' theorem (with $\vec{\nabla} = \left(\frac{\partial}{\partial p_1}\cdots\frac{\partial}{\partial q_J}\right)$. ω was chosen arbitrarily, thus the integrands are related by

$$\begin{aligned}
-\frac{\partial\rho}{\partial t} =& \vec{\nabla}\cdot(\rho\vec{v}) = \sum_{i=1}^{J} \frac{\partial}{\partial p_i}(\rho\dot{p}_i) + \frac{\partial}{\partial q_i}(\rho\dot{q}_i) \\
=& \sum_{i=1}^{J} \frac{\partial\rho}{\partial p_i}\dot{p}_i + \frac{\partial\rho}{\partial q_i}\dot{q}_i + \frac{\partial\dot{p}_i}{\partial p_i}\rho + \frac{\partial\dot{q}_i}{\partial q_i}\rho \\
=& \sum_{i=1}^{J} \frac{\partial\rho}{\partial p_i}\dot{p}_i + \frac{\partial\rho}{\partial q_i}\dot{q}_i - \frac{\partial^2 H}{\partial p_i\partial q_i}\rho + \frac{\partial^2 H}{\partial q_i\partial p_i}\rho \\
=& \sum_{i=1}^{J} \frac{\partial\rho}{\partial p_i}\dot{p}_i + \frac{\partial\rho}{\partial q_i}\dot{q}_i
\end{aligned}$$

Hence,

$$\frac{d\rho}{dt} = \sum_{i=1}^{J} \frac{\partial\rho}{\partial p_i}\dot{p}_i + \frac{\partial\rho}{\partial q_i}\dot{q}_i \;+\; \frac{\partial\rho}{\partial t} = 0$$

□

60 Definition. If $\rho = const.$ *in phase space* then we say that the physical system is in *statistical equilibrium* (or *thermodynamic equilibrium*).

That is, a system is in statistical equilibrium if $\frac{\partial\rho}{\partial p_i} = \frac{\partial\rho}{\partial q_i} = 0$.

The following examples shows that $\frac{\partial\rho}{\partial t}$ can be controlled on a macroscopic level.

61 Example. We imagine a volume of gas consisting of non-interacting particles. By compression we reduce dq_i. Thus, from Liouville's Theorem it follows that dp_i and thus p_i^2 are increased, i.e. the gas is heating up.

10 Canonical transformations

62 Definition. A *canonical transformation*

$$\vec{p},\vec{q} \mapsto \vec{P},\vec{Q}$$

is a transformation of the canonical coordinates that preserves the form Hamilton's equations; that is, we have

$$\mathcal{L} = \sum_i p_i\dot{q}_i - H = \underbrace{\sum_i P_i\dot{Q}_i - K}_{=:\mathcal{L}'} + \frac{dF}{dt}$$

with new Hamiltonian K and new Langrangian $\mathcal{L}'$. F is called *generating function* of the transformation.

The main motivation behind canonical transformations is (a) to simplify the Hamiltonian and (b) to introduce additional cyclic coordinates.

63 Theorem. *The new Lagrangian $\mathcal{L}'$ results in identical trajectories.*

Proof. We have

$$\begin{aligned}
S =& \int_{t_1}^{t_2} Ldt = \int_{t_1}^{t_2} L' + \frac{dF}{dt}dt \\
=& \int_{t_1}^{t_2} L'dt + F(t_2) - F(t_1) \\
=& S' + F(t_2) - F(t_1)
\end{aligned}$$

thus the old action is stationary if and only if the new action is stationary. □

64 Theorem. *The following generating functions define canonical transformations if*

(1) $F_1(\vec{q},\vec{Q},t)$: $\quad p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$

(2) $F_2(\vec{q},\vec{P},t)$: $\quad p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i}$

(3) $F_3(\vec{p},\vec{Q},t)$: $\quad q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$

(4) $F_4(\vec{p},\vec{P},t)$: $\quad q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$

and new Hamiltonian is taken as $K = H + \frac{\partial F_i}{\partial t}$.

Proof. We only prove the statement for the first generating function: by the calculation

$$\begin{aligned}
&\sum_i p_i\dot{q}_i - H \\
=& \sum_i P_i\dot{Q}_i - K + \sum_i\left(\frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i\right) + \frac{\partial F_1}{\partial t}
\end{aligned}$$

and comparison of "coefficients" we conclude that the given conditions are sufficient. □

65 Example (Swapping momenta and coordinates)**.** By theorem 64 the type 1 generating function

$$F_1(\vec{q},\vec{Q}) := \sum_i q_iQ_i$$

leads to

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i, \quad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i$$

that is

$$p_i \mapsto Q_i, \quad -q_i \mapsto P_i$$

66 Example (Point transformatations)**.** The type 2 generating function

$$F_2(\vec{q}, \vec{P}) := \sum_i f_i(q_i, t) P_i$$

leads to

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_i, t), \quad K = H + \frac{\partial F_2}{\partial t}$$

We call transformations of generalized coordinate *point transformations.*

See appendix C for a more involved example.

11 Poisson brackets

67 Definition. The *Poisson bracket* of $u(\vec{p}, \vec{q}, t)$ and $v(\vec{p}, \vec{q}, t)$ is defined by

$$[u, v] := \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

68 Lemma. $[\cdot, \cdot]$ *is a bilinear operator and has the following properties:*

(1) It is invariant with respect to canonical transformations $\vec{p}, \vec{q} \mapsto \vec{P}, \vec{Q}$.

(2) $[u, v] = -[v, u]$

(3) $[q_i, q_j] = [p_i, p_j] = 0, \qquad [q_i, p_j] = \delta_{i,j}$

(4) $[q_i, F] = \frac{\partial F}{\partial p_i}$

(5) $[p_i, F] = -\frac{\partial F}{\partial q_i}$

To be continued.

A Example: Electromagnetic field

Consider a single particle with charge q situated in an electric field $\vec{E}$ and magnetic field $\vec{B}$. We state the following two theorems without proof:

69 Theorem (Maxwell's Equations)**.** *The electromagnetic field satisfies*

(2) $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

(4) $\vec{\nabla} \cdot \vec{B} = 0$

70 Theorem. *(i) If* $\vec{\nabla} \cdot \vec{X} = 0$*, we can find an* $\vec{Y}$ *such that* $\vec{X} = \vec{\nabla} \times \vec{Y}$*.*

(ii) If $\vec{\nabla} \times \vec{X} = 0$*, we can find an* Y *such that* $\vec{X} = \vec{\nabla} Y$*.*

71 Theorem. *We can find a* magnetic vector potential $\vec{A}$ *and an* electric potential Φ *such that*

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \textit{and} \quad \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \Phi$$

Proof. The first claim follows from Maxwell's 4th equation and the preceding theorem.

By Maxwell's 2nd equation we notice

$$\begin{aligned} 0 =& \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \\ =& \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) \end{aligned}$$

and apply the preceding theorem. □

72 Theorem. *The charge's movement in the electromagnetic field is described by the Lagrangian*

$$\mathcal{L} = T - q\Phi + q\vec{A} \cdot \dot{\vec{r}}$$

Proof. By Lorentz and the preceding theorem:

$$\begin{aligned} \vec{F}^e =& q \left(\vec{E} + \dot{\vec{r}} \times \vec{B} \right) \\ =& q \left(-\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t} + \dot{\vec{r}} \times \left(\vec{\nabla} \times \vec{A} \right) \right) \end{aligned}$$

From the Graßmann identity it follows that

$$\begin{aligned} \dot{\vec{r}} \times \left(\vec{\nabla} \times \vec{A} \right) &= \vec{\nabla} \left(\dot{\vec{r}} \cdot \vec{A} \right) - \left(\dot{\vec{r}} \cdot \vec{\nabla} \right) \vec{A} \\ =& \vec{\nabla} \left(\dot{\vec{r}} \cdot \vec{A} \right) - \sum_{i=1}^{3} \dot{x}_i \frac{\partial \vec{A}}{\partial x_i} \\ =& \vec{\nabla} \left(\dot{\vec{r}} \cdot \vec{A} \right) - \frac{d\vec{A}}{dt} + \frac{\partial \vec{A}}{\partial t} \end{aligned}$$

and we arrive at

$$\begin{aligned} \vec{F}^e =& q \left(-\vec{\nabla} \Phi + \vec{\nabla} \left(\dot{\vec{r}} \cdot \vec{A} \right) - \frac{d\vec{A}}{dt} \right) \\ =& q \left(-\vec{\nabla} \left(\Phi - \vec{A} \cdot \dot{\vec{r}} \right) - \frac{d\vec{A}}{dt} \right) \\ =& -\frac{\partial U}{\partial \vec{r}} + \frac{d}{dt} \frac{\partial U}{\partial \dot{\vec{r}}} \end{aligned}$$

with the generalized potential

$$U = q\Phi - q\vec{A} \cdot \dot{\vec{r}}$$

Hence the claim follows. □

73 Corollary. *The Hamiltonian is given by*

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi$$

Proof. By the theorem we have

$$\mathcal{L} = \frac{m}{2}\dot{\vec{r}}^2 - q\Phi + q\vec{A} \cdot \dot{\vec{r}}$$

Hence the conjucated momenta are given by

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}$$

and we follow that

$$\begin{aligned} H =& \vec{p}\dot{\vec{r}} - \mathcal{L} = \frac{m}{2}\dot{\vec{r}}^2 + q\Phi \\ =& \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi \end{aligned}$$

The fact that accounting for the magnet field leads to a modified momentum term in the Hamiltonian is known as the *principle of minimal coupling.* □

B Example: Particle on a rotating wire

Consider a particle (e.g. a pearl) situated on a wire rotating around the origin with $\omega = const.$ We employ two Cartesian coordinates x and y. Realizing that there is just a single degree of freedom we notice that

$$x = \cos(\omega t) \quad \text{and} \quad y = \sin(\omega t)$$

from which find the following holomorphic constraint:

$$y\cos(\omega t) - x\sin(\omega t) = 0$$

By the corollary to theorem 15 we arrive at the Lagrangian

$$\mathcal{L} := \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \lambda\left(y\cos(\omega t) - x\sin(\omega t)\right)$$

which leads to the following ODEs:

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} &= m\ddot{x} + \lambda\sin(\omega t) = 0 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} &= m\ddot{y} - \lambda\cos(\omega t) = 0 \end{aligned}$$

We eliminate λ:

$$\begin{aligned} \lambda =& \frac{m\ddot{y}}{\cos(\omega t)} \\ \Rightarrow& \ddot{x} + \ddot{y}\tan(\omega t) = 0 \end{aligned}$$

From the constraint follows

$$\begin{aligned} y =& x\tan(\omega t) \\ \Rightarrow \ddot{y} =& \ddot{x}\tan(\omega t) + \dot{x}\frac{2\omega}{\cos^2(\omega t)} + x\frac{2\omega^2\tan(\omega t)}{\cos^2(\omega t)} \end{aligned}$$

which we plug into the former differential equation leading to:

$$\ddot{x} + 2\omega\tan(\omega t)\dot{x} + 2\omega^2\tan^2(\omega t)x = 0$$

from which we can determine the trajectory $\binom{x}{y}$.

C Example: Canonical transformation of the harmonic oscillator

Consider the harmonic oscillator with a single degree of freedom whose Hamiltonian is given by

$$H = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 q^2\right)$$

(see example in chapter 9). By theorem 64 the type 1 generating function

$$F_1(q, Q) := \frac{m}{2}\omega q^2 \cot(Q)$$

leads to

$$\begin{aligned} p &= \frac{\partial F_1}{\partial q} = m\omega q\cot(Q) \\ P &= -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2(Q)} \end{aligned}$$

We express p, q purely in terms of P, Q

$$\begin{aligned} q =& \sqrt{\frac{2P}{m\omega}}\sin(Q) \\ \Rightarrow p =& \sqrt{2m\omega P}\cos(Q) \end{aligned}$$

and a quick calculation reveals that the new Hamiltonian is given by

$$K = H = \omega P\cos 2(Q) + \omega P\sin 2(Q) = \omega P$$

We notice that K is cyclic in Q, hence P is conserved and we have

$$P = \frac{H}{\omega}, \quad \dot{Q} = \frac{\partial K}{\partial P} = \omega$$

Thus the motion of the harmonic oscillator is given by

$$\begin{aligned} Q(t) =& \omega t + \varphi \\ \Rightarrow q(t) =& \sqrt{\frac{2P}{m\omega}}\sin(\omega t + \varphi) \end{aligned}$$