THE DIRAC DELTA IS NOT IN L^p

1 Lemma. For each $\epsilon \in (0, \frac{1}{2})$ there exists a function $\varphi_\epsilon \in C_c^\infty((0,2))$ such that:

(i) $\varphi_\epsilon(1) = 1$

(ii) $\operatorname{supp} \varphi_\epsilon = B_\epsilon(1)$

(iii) $||\varphi_\epsilon||_q \leqslant 1$ for all $q \in [1, \infty]$

Proof. Let

$$\varphi : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \begin{cases} \exp(-x^{-1}) & , x > 0 \\ 0 & , x \leqslant 0 \end{cases} \end{cases}$$

We inductively see that $\exp(-\cdot^{-1})$ is infinitely-often differentiable in $\mathbb{R}_{>0}$, and for any $n \in \mathbb{N}_0$ we have

$$\frac{d^n}{dx^n} \exp(-x^{-1}) = \exp(-x^{-1}) p(x^{-1})$$

for a certain polynomial $p \in \mathbb{R}[X]$ (use the chain rule). Hence, Calculus tells us that

$$\lim_{x \to 0+} \frac{d^n}{dx^n} \exp(-x^{-1}) = \lim_{y \to \infty} \exp(-y) p(y) = 0$$

and we conclude that $\varphi \in C^\infty(\mathbb{R})$.

Now let

$$\varphi_\epsilon = (x \mapsto \exp(1) \cdot x) \circ \varphi \circ (x \mapsto 1 - x^2) \circ (x \mapsto \frac{x-1}{\epsilon})|_{(0,2)}$$

which is in $C^\infty((0,2))$ as a composition of infinitely-often differentiable functions.

Note that

$$\varphi_\epsilon(1) = \exp(1) \cdot \varphi(1) = 1$$

which shows (i). And from

$$\begin{aligned} & \varphi_\epsilon(x) = 0 \\ \Leftrightarrow & \left((x \mapsto 1 - x^2) \circ (x \mapsto \frac{x-1}{\epsilon}) \right)(x) \leqslant 0 \\ \Leftrightarrow & |\frac{x-1}{\epsilon}| \geqslant 1 \\ \Leftrightarrow & x \notin B_\epsilon(1) \end{aligned}$$

we see that (ii) holds, and by taking the closure it follows that $\varphi_\epsilon \in C_c^\infty((0,2))$ for $\epsilon \in (0, \frac{1}{2})$.

It is trivial to see that $||\varphi_\epsilon||_\infty = 1$, hence for any $q \in [1, \infty)$ we have

$$\int |\varphi_\epsilon|^p \, d\lambda^1 \overset{(ii)}{=} \int_{1-\epsilon}^{1+\epsilon} |\varphi_\epsilon|^p \, d\lambda^1 \leqslant 2\epsilon ||\varphi_\epsilon||_\infty \leqslant 1$$

(and the integral exists since the integrand is measurable and non-negative) and (iii) holds. ∎

2 Remark. In particular, $\varphi_\epsilon \in L^q((0,2))$ for all $q \in [1,\infty]$.

3 Theorem. Let $p \in [1,\infty]$. Then there exists no $h \in L^p((0,2))$ such that

$$\int_{(0,2)} h \cdot \varphi \cdot d\lambda^1 = \varphi(1) \qquad \forall \varphi \in C_c^\infty((0,2))$$

Proof. Assume there is an $h \in L^p((0,2))$ with the property above.

Let q be the conjugate of p. Further, let φ_ϵ as in the lemma. Then we have

$$\begin{aligned}
1 &= |\varphi_\epsilon(1)| \\
&= |\int h\varphi_\epsilon \, d\lambda^1| \leqslant \int |h| \, |\varphi_\epsilon| \, d\lambda^1 = \int |h1_{B_\epsilon(1)}| \, |\varphi_\epsilon| \, d\lambda^1 \\
&\leqslant ||h1_{B_\epsilon(1)}||_p \cdot ||\varphi_\epsilon||_q \leqslant ||h1_{B_\epsilon(1)}||_p \\
&= \left(\int |h1_{B_\epsilon(1)}|^p d\lambda^1 \right)^{\frac{1}{p}} \xrightarrow{\epsilon \to 0+} 0
\end{aligned}$$

(note the use of Hölder's inequality; the limit follows from the dominated convergence theorem which is applicable as

$$0 \xleftarrow{a.e.} |h1_{B_\epsilon(1)}|^p \leqslant |h|^p$$

and $|h|^p$ is integrable since $h \in L^p((0,2))$.) Contradiction! $\square$