A MOTIVATION OF CHAIN HOMOTOPIES

MICHAEL WALTER

The interval. The cellular chain complex I_* of the interval $[0,1]$ is given by

$$\begin{array}{ccccccccccc} & & \boxed{2} & & \boxed{1} & & \boxed{0} & & \boxed{-1} & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\partial_1^I} & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & & & n & \longmapsto & (n,-n) & & & & \end{array}$$

Note that the map $X \to X \times [0,1], x \mapsto (x,0)$ induces the cellular map i_* defined by

$$i_n : \begin{cases} C_n \to (C_* \otimes I_*)_n \cong C_n \oplus C_n \oplus C_{n-1} \\ \quad x \mapsto (x,0,0) \end{cases}$$

where C_* is the cellular chain complex of X, and similarly for the map j_* induced by $x \mapsto (x,1)$.

Homotopies. A homotopy $h : X \times [0,1] \to Y$ between maps $f, g : X \to Y$ induces a chain map

$$h_* : C_* \otimes I_* \to D_*$$

where C_* and D_* are the cellular chain complexes of X and Y, respectively, and $h_* \circ i_* = f_*$ and $h_* \circ j_* = g_*$.

Chain homotopies. We will take that as the definition. A *chain homotopy* between chain maps $f_*, g_* : C_* \to D_*$ is a chain map

$$h_* : C_* \otimes I_* \to D_*$$

with $h_* \circ i_* = f_*$ and $h_* \circ j_* = g_*$.

Let us simplify this description. The second condition is equivalent to requiring that h_* can be decomposed as

$$h_n = f_n \oplus g_n \oplus H_{n-1} : C_n \oplus C_n \oplus C_{n-1} \cong (C_* \otimes I_*)_n \to D_n.$$

In this picture, the differential of the tensor complex is given by

$$\partial_n(x,y,z) = (\partial_n^C(x) + z, \partial_n^C(y) - z, -\partial_{n-1}^C(z))$$

The requirement that h_* be a chain map is thus equivalent to the condition

$$f_{n-1}(z) - g_{n-1}(z) - H_{n-2}\partial_{n-1}^C(z) = \partial_n^D H_{n-1}(z)$$

since f_* and g_* are already chain maps. After shifting indices we get the following result:

Lemma. A chain homotopy between two chain maps $f_*, g_* : C_* \to D_*$ is the same data as a family of homomorphisms $h_* : C_* \to D_{*+1}$ satisfying

$$f_* - g_* = h_{*-1}\partial_*^C + \partial_{*+1}^D h_*.$$

Date: Last compiled January 28, 2011; last modified January 4, 2010 or later.

E-mail address: michael.walter@gmail.com