A First-Order Logic with First-Class Types

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Java Card DL

- modal logic behind KeY
- based on a typed first-order logic with subtyping, type predicates and casts [Gie05]

$$\forall x : \mathsf{Object}\ .\ x \sqsubseteq \mathsf{Array} \rightarrow \mathsf{length}((\mathsf{Array})x) \geq 0$$

- we focus only on this first-order part

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Java Generics

- classes parametrized by type parameters

```
public class Array<T>
{
  public T last();
}
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- $\text{Array}\langle T \rangle \sqsubseteq \text{Array}\langle ? \rangle \sqsubseteq \text{Object}$
- what is the signature of last?

$$\{\, \text{last}_T : \text{Array}\langle T \rangle \to T \,\}$$

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how to reason about arrays without fixing the element type?

- single signature

$$\mathrm{last} : \mathrm{Array}\langle ?\rangle \to \top$$
$$\mathrm{T} : \mathrm{Array}\langle ?\rangle \to \mathbb{T}$$

  with type of all types $\mathbb{T}$

- need to assert that the return value has proper type

$$\forall a : \mathrm{Array}\langle ?\rangle \;.\; \mathrm{last}(a) \sqsubseteq T(a)$$

  with binary predicate $\sqsubseteq$

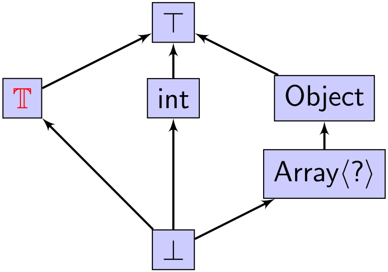
($\rightsquigarrow$ universal types)

Outline

Type Hierarchy

Definition (Type hierarchy)

- set of types $\mathcal{T}$
- subtype relation $\sqsubseteq$
- universal type $\top$ and empty type $\bot$
- greatest lower bounds ($\sqcap$)
- type of all types $\mathbb{T}$



Signature

Definition (Signature)

- predicate, function and variable symbols with types
- predefined symbols:
  - equality $\doteq : \top \times \top$
  - type predicate $\sqsubseteq : \top \times \mathbb{T}$
  - subtype predicate $\sqsubseteq : \mathbb{T} \times \mathbb{T}$

  - type intersection $\sqcap : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}$
  - type constants $T : \rightarrow \mathbb{T}$ (for each type $T \in \mathcal{T}$)
  - casts

Terms and Formulae

Definition (Term of type T)

- v if $v : T$ variable symbol
- $f(t_1, \ldots, t_n)$ if $f : T_1 \times \ldots \times T_n \to T$ function symbol, t_i term of type $T'_i \sqsubseteq T_i$

Definition (Formula)

- $p(t_1, \ldots, t_n)$ if ...
- $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \to \psi$
- $\forall v.\varphi$, $\exists v.\varphi$

Outline

Structure

Definition (Structure)

- domain $\mathcal{D}$
- dynamic typing function $\delta : \mathcal{D} \to \mathcal{T}$

$$\leadsto \mathcal{D}_T := \{x \in \mathcal{D} : \delta(x) \sqsubseteq T\}$$

- interpretation $\mathcal{I}$ of functions and predicates

$$\mathcal{I}(f) : \mathcal{D}_{T_1} \times \ldots \times \mathcal{D}_{T_n} \to \mathcal{D}_T$$
$$\mathcal{I}(p) \sqsubseteq \mathcal{D}_{T_1} \times \ldots \times \mathcal{D}_{T_n}$$

$\leadsto$ value of a term, validity of a formula...

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how about the predefined symbols?

$\rightsquigarrow$ value of a term, validity of a formula...

Interpretation

- $\mathcal{D}_{\mathbb{T}} = \mathcal{T}$
- predefined symbols shall agree with their type hierarchy counterpart:

$$\mathcal{I}(\sqsubseteq\!\!\!\in) \ni (x, T) \;\Leftrightarrow\; x \in \mathcal{D}_T \;\Leftrightarrow\; \delta(x) \sqsubseteq T$$
$$\mathcal{I}(\sqsubseteq) = \sqsubseteq,\ \mathcal{I}(T) = T, \ldots$$

Observation

If the type hierarchy is infinite then the logic has no sound and complete calculus. $\lightning$

Completeness and Compactness

Definition ((Strong) completeness)

$$\mathcal{A} \models \varphi \quad \Rightarrow \quad \mathcal{A} \vdash \varphi$$

Compactness Theorem

Every logic which has a sound and complete calculus is compact: If some set of formulae is not satisfiable then there exists a finite subset which is already not satisfiable.

Reasons for Noncompactness

two obstructions to compactness

1. constant symbols generate domain of $\mathbb{T}$

$$\{\neg(c \doteq T) \ : \ T \in \mathcal{T}\} \quad \lightning$$

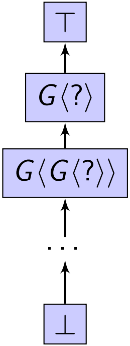
(for infinite $\mathcal{T}$; compare $\mathbb{N}$)

Reasons for Noncompactness

2. non-Noetherian type hierarchies

$$\{x \sqsubseteq G\langle ? \rangle, x \sqsubseteq G\langle G\langle ? \rangle\rangle, \ldots,$$
$$\neg(x \sqsubseteq \bot)\} \quad \lightning$$

(compare induction)



Theorem (Giese)

The logic of [Gie05] has a sound and complete calculus if and only if the type hierarchy is Noetherian.

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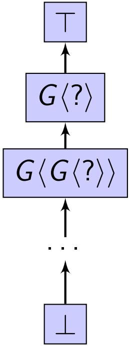

Theorem (Giese)

The logic of [Gie05] has a sound and complete calculus if and only if the type hierarchy is Noetherian.

Interpretation – Modified

- require $\mathcal{D}_{\mathbb{T}}$ to be a type hierarchy that contains $(\mathcal{T}, \sqsubseteq)$
- predefined symbols shall extend their type hierarchy counterparts
- sanity conditions

Theorem

The modified logic has a sound and complete calculus if and only if the type hierarchy is Noetherian.

Outline

Conclusion

- characterized completeness of the logic of [Gie05]
- characterized completeness of first-class types

- first-class types are not useful on their own $\lightning$
  $\rightsquigarrow$ universal types, dependent types

Martin Giese.
A Calculus for Type Predicates and Type Coercion.
In Bernhard Becker, editor, *Proceedings of the 14th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX 2005)*, Lecture Notes in Artificial Intelligence, pages 123–137. Springer, 2005.